Symmetric spaces of the non-compact type : Lie groups Université Joseph Fourier, June 2004

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Contents

1 Introduction

This note is meant to give an introduction to the subjects of Lie groups and equivariant connections on homogeneous spaces. The final goal is the study of the Levi-Civita connection on a symmetric space of the non-compact type. An introduction to the subject of "symmetric space" from the point of view of differential geometry is given in the course of J. Maubon [5].

2 Lie groups and Lie algebras: an overview

In this section, we review the basic notions concerning the Lie groups and the Lie algebras. For a more completed exposition, the reader is invited to consult standard textbook, for example [6], [1] and [3].

Definition 2.1 *A Lie group G is a differentiable manifold*[1] *which is also endowed with a group structure such that the mappings*

$$\begin{array}{rcl} G\times G \longrightarrow & G & , \ (x,y)\longmapsto xy \quad \text{multiplication} \\ G \longrightarrow & G & , \ x\longmapsto x^{-1} \quad \text{inversion} \end{array}$$

are smooth.

We can define in the same way the notion of a *topological group*: it is a topological space[2] which is also endowed with a group structure such that 'multiplication' and 'inversion' mappings are continuous.

The most basic examples of Lie groups are $(\mathbb{R},+)$, $(\mathbb{C}-\{0\},\times)$, and the general linear group $\mathrm{GL}(V)$ of a finite dimensional (real or complex) vector space V. The classical groups like

$$\begin{array}{rcl} \mathrm{SL}(n,\mathbb{R}) & = & \{g\in \mathrm{GL}(\mathbb{R}^n),\ \det(g)=1\}, \\ \mathrm{O}(n,\mathbb{R}) & = & \{g\in \mathrm{GL}(\mathbb{R}^n),\ {}^t gg=\mathrm{Id}_n\}, \\ \mathrm{U}(n) & = & \{g\in \mathrm{GL}(\mathbb{C}^n),\ {}^t\overline{g}g=\mathrm{Id}_n\}, \\ \mathrm{O}(p,q) & = & \{g\in \mathrm{GL}(\mathbb{R}^{p+q}),\ {}^t g\mathrm{I}_{p,q}g=\mathrm{I}_{p,q}\},\ \text{where } \mathrm{I}_{p,q}=\begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q\end{pmatrix} \\ \mathrm{Sp}(\mathbb{R}^{2n}) & = & \{g\in \mathrm{GL}(\mathbb{R}^{2n}),\ {}^t gJg=J\},\ \text{where } J=\begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0\end{pmatrix} \end{array}$$

are all Lie groups. It can be proved by hand, or one can use an old Theorem of E. Cartan.

Theorem 2.2 *Let G be a* closed *subgroup of* $\mathrm{GL}(V)$*. Then G is a embedded submanifold of* $\mathrm{GL}(V)$*, and equipped with this differential structure it is a Lie group.*

The identity element of any group G will be denote by e. We denote the tangent space of the Lie groups G, H, K at the identity element respectively by: $\mathfrak{g}=\mathbf{T}_eG, \quad \mathfrak{h}=\mathbf{T}_eH, \quad \mathfrak{k}=T_eK$.

Example : The tangent space at the identity element of the Lie groups $\mathrm{GL}(\mathbb{R}^n), \mathrm{SL}(n,\mathbb{R}), \mathrm{O}(n,\mathbb{R})$ are respectively

$$\begin{array}{rcl} \mathfrak{gl}(\mathbb{R}^n) & = & \{\text{endomorphism of } \mathbb{R}^n\}, \\ \mathfrak{sl}(n,\mathbb{R}) & = & \{X\in \mathfrak{gl}(\mathbb{R}^n),\ \mathrm{Tr}(X)=0\}, \\ \mathfrak{o}(n,\mathbb{R}) & = & \{X\in \mathfrak{gl}(\mathbb{R}^n),\ {}^tX+X=0\}, \\ \mathfrak{o}(p,q) & = & \{X\in \mathfrak{gl}(\mathbb{R}^n),\ {}^tX\mathrm{Id}_{p,q}+\mathrm{Id}_{p,q}X=0\},\ \text{where } p+q=n. \end{array}$$

[1] All manifolds are second countable.

[2] Here "topological space" means Hausdorff and locally compact.

2.1 Group action

A morphism $\phi : G \to H$ of groups is by definition a map that preserves the product : $\Phi(g_1 g_2) = \Phi(g_1)\phi(g_2)$.

Exercise 2.3 *Show that $\phi(e) = e$ and $\phi(g^{-1}) = \phi(g)^{-1}$.*

Definition 2.4 *An (left)* action of a group G *on a set* M *is a mapping*

$$\alpha : G \times M \longrightarrow M \tag{2.1}$$

such that $\alpha(e, m) = m, \forall m \in M$, and $\alpha(g, \alpha(h, m)) = \alpha(gh, m)$ for all $m \in M$ and $g, h \in G$.

Let $\mathrm{Bij}(M)$ be the group of all bijective maps from M onto M. The conditions on α are equivalent to saying that the map $G \to \mathrm{Bij}(M), g \to \alpha_g$ defined by $\alpha_g(m) = \alpha(g, m)$ is a group morphism .

If G is a Lie (resp. topological) group and M is a manifold (resp. topological space), the action of G on M is said to be smooth (resp. continuous) if the map (2.1) is smooth (resp. continuous). When the notations are understood we will write $g \cdot m$, or simply gm for $\alpha(g, m)$.

A *representation* of a group G on a real vector space (resp. complex) V is a group morphism $\phi : G \to \mathrm{GL}(V)$: the group G acts on V through linear endomorphism.

NOTATION : If $\phi : M \to N$ is a smooth map between differentiable manifolds, we denote by $\mathbf{T}_m\phi : \mathbf{T}_m M \to \mathbf{T}_{\phi(m)} N$ the differential of ϕ at $m \in M$.

2.2 Adjoint representation

Let G be a Lie group and let $\mathfrak{g}$ be the tangent space of G at e. We consider the conjugation action of G on itself defined by

$$c_g(h) = ghg^{-1}, \quad g, h \in G.$$

The mappings $c_g : G \to G$ are smooth and $c_g(e) = e$ for all $g \in G$, so one can consider the differential of c_g at e

$$\mathrm{Ad}(g) = \mathbf{T}_e c_g \ : \mathfrak{g} \to \mathfrak{g}.$$

Since $c_{gh} = c_g \circ c_h$ we have $\mathrm{Ad}(gh) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h)$. That is, the mapping

$$\mathrm{Ad} : G \longrightarrow \mathrm{GL}(\mathfrak{g}) \tag{2.2}$$

is a smooth group morphism which is called the *adjoint representation* of G.

The next step is to consider the differential of the map Ad at e:

$$\mathrm{ad} = \mathbf{T}_e \mathrm{Ad} \ : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathfrak{g}). \tag{2.3}$$

It is the *adjoint representation of* $\mathfrak{g}$. In (2.3), the vector space $\mathfrak{gl}(\mathfrak{g})$ denotes the vector space of all linear endomorphism of $\mathfrak{g}$, and is equal to the tangent space of $\mathrm{GL}(\mathfrak{g})$ at the identity.

Lemma 2.5 . *We have the fundamental relations*

- $\mathrm{ad}(\mathrm{Ad}(g)X) = \mathrm{Ad}(g) \circ \mathrm{ad}(X) \circ \mathrm{Ad}(g)^{-1}$ *for* $g \in G, X \in \mathfrak{g}$.
- $\mathrm{ad}(\mathrm{ad}(Y)X) = \mathrm{ad}(Y) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Y)$ *for* $X, Y \in \mathfrak{g}$.
- $\mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$ *for* $X, Y \in \mathfrak{g}$.

PROOF : Since Ad is a group morphism we have $\mathrm{Ad}(ghg^{-1}) = \mathrm{Ad}(g) \circ \mathrm{Ad}(h) \circ \mathrm{Ad}(g)^{-1}$. If we differentiate this relation at $h = e$ we get the first point, and if we differentiate it at $g = e$ we get the second one.

For the last point consider two smooth curves $a(t), b(s)$ on G with $a(0) = b(0) = e$, $\frac{d}{dt}[a(t)]_{t=0} = X$, and $\frac{d}{dt}[b(t)]_{t=0} = Y$. We will now compute the second derivative $\frac{\partial^2 f}{\partial t \partial s}(0,0)$ of the map $f(t,s) = a(t)b(s)a(t)^{-1}b(s)^{-1}$. Since $f(t,0) = f(0,s) = e$, the term $\frac{\partial^2 f}{\partial t \partial s}(0,0)$ is defined in an intrinxsic manner as an element of $\mathfrak{g}$. For the first partial derivatives we get $\frac{\partial f}{\partial t}(0,s) = X - \mathrm{Ad}(b(s))X$ and $\frac{\partial f}{\partial s}(t,0) = \mathrm{Ad}(a(t))Y - Y$. So $\frac{\partial^2 f}{\partial t \partial s}(0,0) = \mathrm{ad}(X)Y = -\mathrm{ad}(Y)X$. □

Definition 2.6 *If G is a Lie group, one defines a bilinear map,* $[-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *by* $[X,Y]_{\mathfrak{g}} = \mathrm{ad}(X)Y$. *It is the* Lie bracket *of* $\mathfrak{g}$. *The vector space $\mathfrak{g}$ equipped with $[-,-]_{\mathfrak{g}}$ is called the Lie algebra of G. We have the fundamental relations*

- anti symmetry *:* $[X,Y]_{\mathfrak{g}} = -[Y,X]_{\mathfrak{g}}$
- Jacobi identity *:* $\mathrm{ad}([Y,X]_{\mathfrak{g}}) = \mathrm{ad}(Y) \circ \mathrm{ad}(X) - \mathrm{ad}(X) \circ \mathrm{ad}(Y)$.

On $\mathfrak{gl}(\mathfrak{g})$, a direct computation shows that $[X,Y]_{\mathfrak{gl}(\mathfrak{g})} = XY - YX$. So the Jacobi identity can be rewritten as $\mathrm{ad}([X,Y]_{\mathfrak{g}}) = [\mathrm{ad}(X), \mathrm{ad}(Y)]_{\mathfrak{gl}(\mathfrak{g})}$ or equivalently as

$$[X,[Y,Z]_{\mathfrak{g}}]_{\mathfrak{g}} + [Y,[Z,X]_{\mathfrak{g}}]_{\mathfrak{g}} + [Z,[X,Y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0 \quad \text{for all } X, Y, Z \in \mathfrak{g}. \tag{2.4}$$

Definition 2.7 • *A* Lie algebra $\mathfrak{g}$ *is a real vector space equipped with the antisymmetric bilinear map* $[-,-]_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *satisfying the Jacobi identity.*

• *A linear map* $\phi : \mathfrak{g} \to \mathfrak{h}$ *between two Lie algebras is a* morphism *of Lie algebras if*

$$\phi([X,Y]_{\mathfrak{g}}) = [\phi(X), \phi(Y)]_{\mathfrak{h}}. \tag{2.5}$$

Remark 2.8 *We have defined the notion of real Lie algebra. The definitions goes through on any field k, in particular when $k = \mathbb{C}$ we speak of complex Lie algebras. For example, if $\mathfrak{g}$ is a real Lie algebra, the complexified vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ inherits a canonical structure of complex Lie algebra.*

The map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the typical example of a morphism of Lie algebras. This example generalizes as follows.

Lemma 2.9 *Consider a smooth morphism $\Phi : G \to H$ between two Lie groups. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be its differential at e. Then:*

- *The map ϕ is Φ-equivariant: $\phi \circ \mathrm{Ad}(g) = \mathrm{Ad}(\Phi(g)) \circ \phi$.*
- *ϕ is a morphism of Lie algebras.*

The proof works as in Lemma 2.5.

Example : If G is a closed subgroup of $\mathrm{GL}(V)$, the inclusion $\mathfrak{g} \hookrightarrow \mathfrak{gl}(V)$ is a morphism of Lie algebra. In other words, if $X, Y \in \mathfrak{g}$ then $[X, Y]_{\mathfrak{gl}(V)} = XY - YX$ belongs to $\mathfrak{g}$ and corresponds to the Lie bracket $[X, Y]_{\mathfrak{g}}$.

2.3 Vectors fields and Lie bracket

Here we review the typical example of Lie bracket : those of vectors fields.

Let M be a smooth manifold. We denote by $\mathrm{Diff}(M)$ the group formed by the diffeomorphism of M, and by $\mathrm{Vect}(M)$ the vector space of smooth vectors fields. Even if $\mathrm{Diff}(M)$ is not a Lie group (it's not finite dimensional), many aspects discuss earlier apply here, with $\mathrm{Vect}(M)$ in the role of the Lie algebra of $\mathrm{Diff}(M)$. If $a(t)$ is a smooth curve in $\mathrm{Diff}(M)$ passing through the identity at $t = 0$, the derivative $V = \frac{d}{dt}[a]_{t=0}$ is a vectors field on M.

The "adjoint" action of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ is defined as follows. If $V = \frac{d}{dt}[a]_{t=0}$ one takes $\mathrm{Ad}(g)V = \frac{d}{dt}[g \circ a \circ g^{-1}]_{t=0}$ for every $g \in \mathrm{Diff}(M)$. The definition of Ad extends to any $V \in \mathrm{Vect}(M)$ through the following expression

$$\mathrm{Ad}(g)V|_m = \mathbf{T}_{g^{-1}m}(g)(V_{g^{-1}m}), \qquad m \in M. \tag{2.6}$$

We can now define the adjoint action by differentiating (2.6) at the identity. If $W = \frac{d}{dt}[b]_{t=0}$ and $V \in \mathrm{Vect}(M)$, we take

$$\mathrm{ad}(W)V|_m = \frac{d}{dt}\left[\mathbf{T}_{b(t)^{-1}m}(b(t))(V_{b(t)^{-1}m})\right]_{t=0}, \qquad m \in M. \tag{2.7}$$

If we take any textbook on differential geometry we see that $\mathrm{ad}(W)V = -[W, V]$, where $[-,-]$ is the usual Lie bracket on $\mathrm{Vect}(M)$. To explain why we get this minus sign, consider the group morphism

$$\begin{aligned} \Phi : \mathrm{Diff}(M) &\longrightarrow \mathrm{Aut}(\mathcal{C}^\infty(M)) \\ g &\longmapsto \underline{g} \end{aligned} \tag{2.8}$$

defined by $\underline{g} \cdot f(m) = f(g^{-1}m)$ for $f \in \mathcal{C}^\infty(M)$. Here $\mathrm{Aut}(\mathcal{C}^\infty(M))$ is the group of automorphism of the algebra $\mathcal{C}^\infty(M)$. If $b(t)$ is a smooth curve in $\mathrm{Aut}(\mathcal{C}^\infty(M))$ passing through the identity at $t = 0$, the derivative $u = \frac{d}{dt}[b]_{t=0}$ belongs to the vector space $\mathrm{Der}(\mathcal{C}^\infty(M))$ of derivations of $\mathcal{C}^\infty(M)$: $u : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is a linear map and $u(fg) = u(f)g + fu(g)$. So the Lie algebra of $\mathrm{Aut}(\mathcal{C}^\infty(M))$ as a natural identification with $\mathrm{Der}(\mathcal{C}^\infty(M))$ equipped with the Lie bracket: $[u, v]_{\mathrm{Der}} = u\circ v - v\circ u$, for $u, v \in \mathrm{Der}(\mathcal{C}^\infty(M))$.

Let $\mathrm{Vect}(M) \xrightarrow{\sim} \mathrm{Der}(\mathcal{C}^\infty(M))$, $V \mapsto \widetilde{V}$ be the canonical identification defined by $\widetilde{V} f(m) = \langle df_m, V_m\rangle$ for $f \in \mathcal{C}^\infty(M)$ and $V \in \mathrm{Vect}(M)$.

For the differential at the identity of Φ we get

$$d\Phi(V) = -\widetilde{V}, \quad \text{for} \quad V \in \mathrm{Vect}(M). \tag{2.9}$$

Since $d\Phi$ is an algebra morphism we have $-\widetilde{\mathrm{ad}(V)W} = [\widetilde{V}, \widetilde{W}]_{\mathrm{Der}}$. Hence we see that $[V, W] = -\mathrm{ad}(V)W$ is the traditional Lie bracket on $\mathrm{Vect}(M)$ defined by posing $\widetilde{[V, W]} = \widetilde{V} \circ \widetilde{W} - \widetilde{V} \circ \widetilde{W}$.

2.4 Group actions and Lie bracket

Let M be a differentiable manifold equipped with a smooth action of a Lie group G. We can specialize (2.8) to a group morphism $G \to \mathrm{Aut}(\mathcal{C}^\infty(M))$. Its differential at the identity defines a map $\mathfrak{g} \to \mathrm{Der}(\mathcal{C}^\infty(M)) \xrightarrow{\sim} \mathrm{Vect}(M)$, $X \to X_M$ by $X_M|_m = \frac{d}{dt}[a(t)^{-1} \cdot m]_{t=0},\ m \in M$. Here $a(t)$ is a smooth curve on G such that $X = \frac{d}{dt}[a]_{t=0}$. This mapping is a morphism of Lie algebras:

$$[X, Y]_M = [X_M, Y_M]. \tag{2.10}$$

Example : Consider the actions of translations R, L of a Lie group G on itself:

$$R(g)h = hg^{-1}, \quad L(g)h = gh \quad \text{for } g, h \in G. \tag{2.11}$$

Theses actions defines vectors field X^L, X^R on G for any $X \in \mathfrak{g}$, and (2.10) reads

$$[X, Y]^L = [X^L, Y^L], \quad [X, Y]^R = [X^R, Y^R].$$

Theses equations can be used to define the Lie bracket on $\mathfrak{g}$. Consider the subspaces $V^L = \{X^L,\ X \in \mathfrak{g}\}$ and $V^R = \{X^R,\ X \in \mathfrak{g}\}$ of $\mathrm{Vect}(G)$. First we see that V^L (resp. V^R) coincides with the subspace of $\mathrm{Vect}(G)^R$ (resp. $\mathrm{Vect}(G)^L$) formed by the vectors fields invariant by the R-action of G (resp. L-action of G). Second we see that the subspaces $\mathrm{Vect}(G)^R$ and $\mathrm{Vect}(G)^L$ are invariant under the Lie bracket of $\mathrm{Vect}(G)$. Then for any $X, Y \in \mathfrak{g}$, the vectors field $[X^L, Y^L] \in \mathrm{Vect}(G)^R$, so there exist a unique $[X, Y] \in \mathfrak{g}$ such that $[X, Y]^L = [X^L, Y^L]$.

2.5 Exponential map

Consider the usual exponential map $\mathrm{e} : \mathfrak{gl}(V) \to \mathrm{GL}(V)$: $\mathrm{e}^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$. We have the fundamental property

Proposition 2.10 • *For any $A \in \mathfrak{gl}(V)$, the map $\phi_A : \mathbb{R} \to \mathrm{GL}(V), t \mapsto \mathrm{e}^{tA}$ is a smooth Lie group morphism with $\frac{d}{dt}[\phi_A]_{t=0} = A$.*

• *If $\phi : \mathbb{R} \to \mathrm{GL}(V)$ is a smooth Lie group morphism we have $\phi = \phi_A$ for $A = \frac{d}{dt}[\phi]_{t=0}$.*

Now, we will see that an exponential map together with Proposition 2.10 exists on all Lie group.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. For any $X \in \mathfrak{g}$ we consider the vectors field $X^R \in \mathrm{Vect}(G)$ defined by $X^R|_g = \frac{d}{dt}[ga(t)]_{t=0},\ g \in G$. Here $a(t)$ is a smooth curve on G such that $X = \frac{d}{dt}[a]_{t=0}$. The vectors fields X^R are invariant under the left translations, that is

$$\mathbf{T}_g(L(h))(X^R_g) = X^R_{hg}, \quad \text{for } g, h \in G. \tag{2.12}$$

We consider now the flow of the vectors field X^R. For any $X \in \mathfrak{g}$ we consider the differential equation

$$\begin{aligned} \frac{\partial}{\partial t}\phi(t, g) &= X^R(\phi(t, g)) \\ \phi(0, g) &= g. \end{aligned} \tag{2.13}$$

where $t \in \mathbb{R}$ belongs to an interval containing 0, and $g \in G$. Classical results assert that for any $g_0 \in G$ (2.13) admits a unique solution ϕ^X defined on $]-\varepsilon, \varepsilon[\times \mathcal{U}$ where $\varepsilon > 0$ is small enough and $\mathcal{U}$ is a neighborhood of g_0. Since X^R is invariant under the left translations we have

$$\phi^X(t, g) = g\phi^X(t, e). \tag{2.14}$$

The map $t \to \phi^X(t, -)$ is a 1-parameter subgroup of (local) diffeomorphisms of M: $\phi^X(t + s, m) = \phi^X(t, \phi^X(s, m))$ for t, s small enough. Eq. (2.14) give then

$$\phi^X(t + s, e) = \phi^X(t, e)\phi^X(s, e) \quad \text{for } t, s \text{ small enough.} \tag{2.15}$$

The map $t \mapsto \phi^X(t,e)$ initially defined on an interval $]-\varepsilon,\varepsilon[$ can be extended on $\mathbb{R}$ thanks to (2.15). For any $t \in \mathbb{R}$ take $\Phi^X(t,e) = \phi^X(\frac{t}{n},e)^n$ where n is an integer large enough so that $|\frac{t}{n}| < \varepsilon$. It is not difficult to see that our definition make sense and that $\mathbb{R} \to G$, $t \mapsto \Phi^X(t,e)$ is a Lie group morphism. Finally we have proved that the vectors field X^R is completed: its flow is defined on $\mathbb{R} \times G$.

Definition 2.11 *For each $X \in \mathfrak{g}$, the element $\exp_G(X) \in G$ is defined as $\Phi^X(1,e)$. The mapping $\mathfrak{g} \to G$, $X \mapsto \exp_G(X)$ is called the exponential mapping from $\mathfrak{g}$ into G.*

Proposition 2.12 • $\exp_G(tX) = \Phi^X(t,e)$ *for each $t \in \mathbb{R}$.*
• $\exp_G : \mathfrak{g} \to G$ *is $\mathcal{C}^\infty$ and $\mathbf{T}_e \exp_G$ is the identity map.*

Proof : Let $s \neq 0$ in $\mathbb{R}$. The maps $t \to \Phi^X(t,e)$ and $t \to \Phi^{sX}(t\frac{X}{s},e)$ are both solutions of the differential equation (2.13): so there are equal and $a)$ is proved by taking $t = s$. To proved $b)$ consider the vectors field V on $\mathfrak{g} \times G$ defined by $V(X,g) = (X^R(g),0)$. It is easy to see that the flow Φ^V of the vectors field V satisfies $\Phi^V(t,X,g) = (g\exp_G(tX),X)$, for $(t,X,g) \in \mathbb{R} \times \mathfrak{g} \times G$. Since Φ^V is smooth (a general property concerning the flows), the exponential map is smooth. □

Proposition 2.10 take now the following form.

Proposition 2.13 *If $\phi : \mathbb{R} \to G$ is a ($\mathcal{C}^\infty$) one parameter subgroup, we have $\phi(t) = \exp_G(tX)$ with $X = \frac{d}{dt}[\phi]_{t=0}$.*

Proof : If we differentiate the relation $\phi(t+s) = \phi(t)\phi(s)$ at $s = 0$, we see that ϕ satisfies the differential equation $(*)$ $\frac{d}{dt}[\phi]_t = X^R(\phi(t))$, where $X = \frac{d}{dt}[\phi]_{t=0}$. Since $t \to \Phi^X(t,e)$ is also solution of $(*)$, and $\Phi^X(0,e) = \phi(0) = e$, we have $\phi = \Phi^X(-,e)$. □

We give now some easy consequences of Proposition 2.13.

Proposition 2.14 • *If $\rho : G \to H$ is a morphism of Lie groups and $d\rho : \mathfrak{g} \to \mathfrak{h}$ is the corresponding morphism of Lie algebras, we have $\exp_H \circ d\rho = \rho \circ \exp_G$.*
• *For $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ we have $\mathrm{Ad}(\exp_G(X)) = \mathrm{e}^{\mathrm{ad}(X)}$.*
• $\exp_G : \mathfrak{g} \to G$ *is G-equivariant: $\exp_G(\mathrm{Ad}(g)X) = g\exp_G(X)g^{-1}$.*
• *If $[X,Y] = 0$, then $\exp_G(X)\exp_G(Y) = \exp_G(Y)\exp_G(X) = \exp_G(X+Y)$.*

Proof : We use in each case the same type of proof. We consider two 1-parameters subgroup $\Phi_1(t)$ and $\Phi_2(t)$. After we verify that $\frac{d}{dt}[\Phi_1]_{t=0} =$

$\frac{d}{dt}[\Phi_2]_{t=0}$, and from Proposition 2.13 we conclude that $\Phi_1(t) = \Phi_2(t),\ \forall t \in \mathbb{R}$. The relation that we are looking for is $\Phi_1(1) = \Phi_2(1)$.

For the first point, we take $\Phi_1(t) = \exp_H(td\rho(X))$ and $\Phi_2(t) = \rho \circ \exp_G(tX)$: for the second point we take $\rho = \mathrm{Ad}$, and for the third one we take $\Phi_1(t) = \exp_G(t\mathrm{Ad}(g)X)$ and $\Phi_2(t) = g\exp_G(tX)g^{-1}$.

From the second and third point we have $\exp_G(X)\exp_G(Y)\exp_G(-X) = \exp_G(\mathrm{e}^{\mathrm{ad}(X)}Y)$. Hence $\exp_G(X)\exp_G(Y)\exp_G(-X) = \exp_G(Y)$ if $\mathrm{ad}(X)Y = 0$. We consider after the 1-parameters subgroups $\Phi_1(t) = \exp_G(tX)\exp_G(tY)$ and $\Phi_2(t) = \exp_G(t(X+Y))$ to prove the second equality of the last point. $\square$

Exercise 2.15 *We consider the Lie group* $\mathrm{SL}(2,\mathbb{R})$ *with Lie algebra* $\mathrm{sl}(2,\mathbb{R}) = \{X \in \mathrm{End}(\mathbb{R}^2),\ \mathrm{Tr}(X) = 0\}$. *Show that the image of the exponential map* $\exp : \mathrm{sl}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})$ *is equal to* $\{g \in \mathrm{SL}(2,\mathbb{R}),\ \mathrm{Tr}(g) \geq -2\}$

Remark 2.16 *The map* $\exp_G : \mathfrak{g} \to G$ *is in general not surjective. Nevertheless the set* $U = \exp_G(\mathfrak{g})$ *is a neighborhood of the identity, and* $U = U^{-1}$. *The subgroup of* G *generated by* U, *which is equal to* $\cup_{n\geq 1}U^n$, *is then a connected open subgroup of* G. *Hence* $\cup_{n\geq 1}U^n$ *is equal to the connected component of the identity, usually denoted* G^o.

Exercise 2.17 *For any Lie group* G, *show that* $\exp_G(X)\exp_G(Y) = \exp_G(X + Y + \frac{1}{2}[X,Y] + o(|X|^2 + |Y|^2))$ *in a neighborhood of* $(0,0) \in \mathfrak{g}^2$. *Afterward show that*

$$\lim_{n\to\infty} (\exp_G(X/n)\exp_G(Y/n))^n = \exp_G(X+Y) \quad \text{and}$$

$$\lim_{n\to\infty} (\exp_G(X/n)\exp_G(Y/n)\exp_G(-X/n)\exp_G(-Y/n))^{n^2} = \exp([X,Y]).$$

2.6 Lie subgroups and Lie subalgebras

Before giving the precise definition of a *Lie subgroup*, we look at the infinitesimal side. A *Lie subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ stable under the Lie bracket : $[X,Y]_{\mathfrak{g}} \in \mathfrak{h}$ whenever $X, Y \in \mathfrak{h}$.

We have a natural extension of Theorem 2.2

Theorem 2.18 *Let* H *be a* closed *subgroup of a Lie group* G. *Then* H *is a imbedded submanifold of* G, *and equipped with this differential structure it is a Lie group. The Lie algebra of* H, *which is equal to* $\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for all } t \in \mathbb{R}\}$, *is a subalgebra of* $\mathfrak{g}$.

PROOF : The two limits given in the exercise 2.17 show that $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ (we use here that H is closed). Let $\mathfrak{a}$ be any supplementary subspace of $\mathfrak{h}$: one shows that $(\exp(Y) \in H) \Longrightarrow (Y = e)$ if $Y \in \mathfrak{a}$ belongs to a small neighborhood of 0 in $\mathfrak{a}$. Now we consider the map $\phi : \mathfrak{h} \oplus \mathfrak{a} \to G$ given by $\phi(X + Y) = \exp_G(X) \exp_G(Y)$. Since $\mathbf{T}_e\phi$ is the identity map, ϕ defines a smooth diffeomorphism $\phi|_{\mathcal{V}}$ from a neighborhood $\mathcal{V}$ of $0 \in \mathfrak{g}$ to a neighborhood $\mathcal{W}$ of e in G. If $\mathcal{V}$ is small enough we see that ϕ map $\mathcal{V} \cap \{Y = 0\}$ onto $\mathcal{W} \cap H$, hence H is a submanifold near e. Near any point $h \in H$ we use the map $\phi_h : \mathfrak{h} \oplus \mathfrak{a} \to G$ given by $\phi_h(Z) = h\phi(Z)$: we prove in the same way that H is a submanifold near h. Finally H is an imbedded submanifold of G. We now look to the group operations $m_G : G \times G \to G$ (multiplication), $i_G : G \to G$ (inversion) and their restrictions $m_G|_{H\times H} : H \times H \to G$ and $i_G|_H : H \to G$ which are smooth maps. Here we are interested in the group operations m_H and i_H of H. Since $m_G|_{H\times H}$ and $i_G|_H$ are smooth we have the equivalence:

$$m_H \text{ and } i_H \text{ are smooth} \Longleftrightarrow m_H \text{ and } i_H \text{ are continuous.}$$

The fact that m_H and i_H are continuous follows easily from the fact that $m_G|_{H\times H}$ and $i_G|_H$ are continuous and that H is closed. □

Theorem 2.18 has the following important corollary

Corollary 2.19 *If $\phi : G \to H$ is a* continuous *group morphism between two Lie groups, then ϕ is smooth.*

PROOF :Consider the graph $L \subset G \times H$ of the map ϕ : $L = \{(g, h) \in G \times H \mid h = \phi(g)\}$. Since ϕ is a continuous L is a closed subgroup of $G \times H$. Following Theorem 2.18, L is an imbedded submanifold of $G \times H$. Consider now the morphism $p_1 : L \to G$ (resp. $p_2 : L \to H$) equals respectively to the composition of the inclusion $L \hookrightarrow G \times H$ with the projection $G \times H \to G$ (resp. $G \times H \to H$): p_1 and p_2 are smooth, p_1 is bijective, and $\phi = p_2 \circ (p_1)^{-1}$. Since $(p_1)^{-1}$ is smooth (see Exercise 2.24), the map ϕ is smooth. □

We have just seen the archetype of a Lie subgroup : a closed subgroup of a lie group. But this notion is too restrictive.

Definition 2.20 *(H, ϕ) is a* Lie subgroup *of a Lie group G if*

- *H is a Lie group,*
- *$\phi : H \to G$ is a group morphism,*
- *$\phi : H \to G$ is a one-to-one immersion.*

In the next example we consider the 1-parameter Lie subgroups of $S^1 \times S^1$: either they are closed or dense.

EXAMPLE : Consider the group morphisms $\phi_\alpha : \mathbb{R} \to S^1 \times S^1$, $\phi_\alpha(t) = (\mathrm{e}^{it}, \mathrm{e}^{i\alpha t})$, defined for $\alpha \in \mathbb{R}$. Then :

• If $\alpha \notin \mathbb{Q}$, $\mathrm{Ker}(\phi_\alpha) = 0$ and $(\mathbb{R}, \phi_\alpha)$ is a Lie subgroup of $S^1 \times S^1$ which is dense.

• If $\alpha \in \mathbb{Q}$, $\mathrm{Ker}(\phi_\alpha) \neq 0$, and ϕ_α factorizes in a smooth morphism $\widetilde{\phi_\alpha} : S^1 \to S^1 \times S^1$. Here $\phi_\alpha(\mathbb{R})$ is a closed subgroup of $S^1 \times S^1$ diffeomorphic to the Lie subgroup $(S^1, \widetilde{\phi_\alpha})$.

Let (H, ϕ) is a Lie subgroup of G, and let $\mathfrak{h}, \mathfrak{g}$ be their respective Lie algebras. Since ϕ is an immersion, the differential at the identity, $d\phi : \mathfrak{h} \to \mathfrak{g}$, is an injective morphism of Lie algebras : $\mathfrak{h}$ is isomorphic with the subalgebra $d\phi(\mathfrak{h})$ of $\mathfrak{g}$. In practice we often "forget" ϕ in our notations, and speak of a Lie subgroup $H \subset G$ with Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. We have to be careful : when H is not closed in G, the topology of H *is not* the induced topology.

We state now the fundamental

Theorem 2.21 *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be a subalgebra. Then there exists a unique connected Lie subgroup H of G with Lie algebra equal to $\mathfrak{h}$. Moreover H is generated by $\exp_G(\mathfrak{h})$, where $\exp_G$ is the exponential map of G.*

The proof uses Frobenius Theorem (see [6][Theorem 3.19]). This Theorem has an important corollary.

Corollary 2.22 *Let G, H be two connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$. Let $\phi : \mathfrak{g} \to \mathfrak{h}$ be a morphism of Lie algebras. If G is simply connected there exits a (unique) Lie group morphism $\Phi : G \to H$ such that $d\Phi = \phi$.*

PROOF : Consider the graph $\mathfrak{l} \subset \mathfrak{g} \times \mathfrak{h}$ of the map ϕ : $\mathfrak{l} := \{(X, Y) \in \mathfrak{g} \times \mathfrak{h} \,|\, \phi(X) = Y\}$. Since ϕ is morphism of Lie algebras $\mathfrak{l}$ is a Lie subalgebra of $\mathfrak{g} \times \mathfrak{h}$. Let (L, ψ) be the connected Lie subgroup of $G \times H$ associated to $\mathfrak{l}$. Consider now the morphism $p_1 : L \to G$ (resp. $p_2 : L \to H$) equals respectively to the composition of $\phi : L \to G \times H$ with the projection $G \times H \to G$ (resp. $G \times H \to H$). The group morphism $p_2 : L \to G$ is onto with a discrete kernel since G is connected and $dp_2 : \mathfrak{l} \to \mathfrak{g}$ is an isomorphism. Hence $p_2 : L \to G$ is a covering map (see Exercise 2.24). Since G is simply connected, this covering map is a diffeomorphism. The group morphism $p_1 \circ (p_2)^{-1} : G \to H$ answers to the question. □

EXAMPLE : The Lie group SU(2) is composed by the 2×2 complex matrices of the form $\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$. Hence SU(2) is simply connected since it is diffeomorphic to the 3-dimensional sphere. Since SU(2) is a maximal compact subgroup of $\mathrm{SL}(2,\mathbb{C})$, the Cartan decomposition (see Section 3.4) tells us that $\mathrm{SL}(2,\mathbb{C})$ is also simply connected.

A subset A of a topological space M is *path-connected* if any points $a, b \in A$ can be joined by a continuous path $\gamma : [0,1] \to M$ with $\gamma(t) \in A$ for all $t \in [0,1]$. Any connected Lie subgroup of a Lie group is path-connected. We have the following characterization of the connected Lie subgroups.

Theorem 2.23 *Let G be a Lie group, and let H be a* path-connected *subgroup of G. Then H is a Lie subgroup of G.*

Exercise 2.24 *Let $\rho : G \to H$ be a smooth morphism of Lie groups, and let $d\rho : \mathfrak{g} \to \mathfrak{h}$ be the corresponding morphism of Lie algebras.*

• *Show that* $\mathrm{Ker}(\rho) := \{g \in G \,|\, \rho(g) = e\}$ *is a closed (normal) subgroup with lie algebra* $\mathrm{Ker}(d\rho) := \{X \in \mathfrak{g} \,|\, d\rho(X) = 0\}$.

• *If* $\mathrm{Ker}(d\rho) = 0$*, show that* $\mathrm{Ker}(\rho)$ *is discrete in G. If furthermore ρ is onto, then show that ρ is a covering map.*

• *If $\rho : G \to H$ is bijective, then show that ρ^{-1} is smooth.*

2.7 Ideals

A subalgebra $\mathfrak{h}$ of a Lie algebra is called an *ideal* in $\mathfrak{g}$ if $[X,Y]_{\mathfrak{g}} \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$: in other words $\mathfrak{h}$ is a stable subspace of $\mathfrak{g}$ under the endomorphism $\mathrm{ad}(Y), Y \in \mathfrak{g}$. A Lie subgroup H of the Lie group G is a *normal subgroup* if $gHg^{-1} \subset H$ for all $g \in G$.

Proposition 2.25 *Let H be the connected Lie subgroup of G associated to the subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The following assertions are equivalent.*

1) *H is a normal subgroup of G^o.*

2) *$\mathfrak{h}$ is an ideal of $\mathfrak{g}$.*

PROOF : **1) $\Longrightarrow$ 2)**. Let $X \in \mathfrak{h}$ and $g \in G^o$. For every $t \in \mathbb{R}$, the element $g \exp_G(tX) g^{-1} = \exp_G(t\mathrm{Ad}(g)X)$ belongs to H: if we take the derivative at $t = 0$ we get $(*)$ $\mathrm{Ad}(g)X \in \mathfrak{h}$, $\forall g \in G^o$. If we take the differential of $(*)$ at $g = e$ we have $\mathrm{ad}(Y)X \in \mathfrak{h}$ whenever $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$.

2) $\Longrightarrow$ 1). If $X \in \mathfrak{h}$ and $Y \in \mathfrak{g}$, we have $\exp_G(Y)\exp_G(X)\exp_G(Y)^{-1} = \exp_G(e^{\mathrm{ad}Y}X) \in H$. Since H is generated by $\exp_G(\mathfrak{h})$, we have $\exp_G(Y) H \exp_G(Y)^{-1} \subset H$ for all $Y \in \mathfrak{g}$ (see Remark 2.16 and Proposition 2.21). Since $\exp_G(\mathfrak{g})$ generates G^o we have finally that $gHg^{-1} \subset H$ for all $g \in G^o$. □

EXAMPLES OF IDEALS : The *center* of $\mathfrak{g}$: $Z_{\mathfrak{g}} := \{X \in \mathfrak{g} \mid [X,\mathfrak{g}] = 0\}$. The *commutator ideal* $[\mathfrak{g},\mathfrak{g}]$. The kernel $\ker(\phi)$ of a morphism of lie algebra $\phi : \mathfrak{g} \to \mathfrak{h}$.

We can associate to any Lie algebra $\mathfrak{g}$ two sequences $\mathfrak{g}_i, \mathfrak{g}^i$ of ideals of $\mathfrak{g}$. The *commutator series* of $\mathfrak{g}$ is the non increasing sequence of ideals $\mathfrak{g}^i$ with

$$\mathfrak{g}^0 = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}^i]. \tag{2.16}$$

The *lower central series* of $\mathfrak{g}$ is the non increasing sequence of ideals $\mathfrak{g}_i$ with

$$\mathfrak{g}_0 = \mathfrak{g} \quad \text{and} \quad \mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]. \tag{2.17}$$

Exercise 2.26 *Show that the $\mathfrak{g}_i, \mathfrak{g}^i$ are ideals of $\mathfrak{g}$.*

Definition 2.27 *We say that $\mathfrak{g}$ is*
- solvable *if $\mathfrak{g}^i = 0$ for i large enough,*
- nilpotent *if $\mathfrak{g}_i = 0$ for i large enough,*
- abelian *if $[\mathfrak{g}, \mathfrak{g}] = 0$.*

Exercise 2.28 *Let V be a finite dimensional vector space, and let $\{0\} = V_0 \subset V_1 \subset \cdots V_n = V$ be a strictly increasing sequence of subspaces. Let $\mathfrak{g}$ be the Lie subalgebra of $\mathfrak{gl}(V)$ defined by $\mathfrak{g} = \{X \in \mathfrak{gl}(V) \mid X(V_{k+1}) \subset V_k\}$.*
- *Show that the Lie algebra $\mathfrak{g}$ is nilpotent.*
- *Suppose now that $\dim V_k = k$ for any $k = 0, \ldots, n$. Show then that the Lie algebra of $\mathfrak{h} = \{X \in \mathfrak{gl}(V) \mid X(V_k) \subset V_k\}$ is solvable.*

Exercise 2.29 *For a group G, the subgroup generated by the commutators $ghg^{-1}h^{-1}$, $g, h \in G$ is the* derived subgroup*, and is denoted by G'.*
- *Show that G' is a normal subgroup of G.*
- *If G is a connected Lie group, show that G' is* the *connected Lie subgroup associated to the ideal $[\mathfrak{g}, \mathfrak{g}]$.*

Exercise 2.30 • *For any Lie group G, show that its center $Z_G := \{g \in G \mid hg = hg \ \forall h \in G\}$ is a closed normal subgroup with Lie algebra $Z_{\mathfrak{g}} := \{X \in \mathfrak{g} \mid [X, Y] = 0, \ \forall Y \in \mathfrak{g}\}$.*
- *Show that a lie algebra $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is solvable.*
- *Let $\mathfrak{h}$ be the Lie algebra of the group H defined in Exercise 2.28. Show that $[\mathfrak{h}, \mathfrak{h}]$ is nilpotent, and that $\mathfrak{h}$ is not nilpotent.*

2.8 Group actions and quotients

Let M be a set equipped with an action of group G. For each $m \in M$ the *G-orbit through m* is defined as the subset

$$G \cdot m = \{g \cdot m \mid g \in G\}. \tag{2.18}$$

For each $m \in M$, the *stabilizer group* at m is

$$G_m = \{g \in G \mid g \cdot m = m\}. \tag{2.19}$$

The G-action is *free* if $G_m = \{e\}$ for all $m \in M$. The G-action is *transitive* if $G \cdot m = M$ for some $m \in M$. The set-theoretic quotient M/G corresponds to the quotient of M by the equivalence relation $m \sim n \iff G \cdot m = G \cdot n$. Let $\pi : M \to M/G$ be the canonical projection.

TOPOLOGICAL SIDE : Suppose now that M is a topological space equipped with a continuous action of a topological[3] group G. Note that in this situation the stabilizers G_m are closed in G. We define for any subsets A, B of M the set

$$G_{A,B} = \{g \in G \mid (g \cdot A) \cap B \neq \emptyset\}.$$

Exercise 2.31 *Show that $G_{A,B}$ is closed in G when A, B are compact in M.*

We take on M/G the *quotient topology*: $\mathcal{V} \subset M/G$ is open if $\pi^{-1}(\mathcal{V})$ is open in M. It is the smallest topology that makes π continuous. Note that $\pi : M \to M/G$ is then an *open map* : if $\mathcal{U}$ is open in M, $\pi^{-1}(\pi(\mathcal{U})) = \cup_{g \in G} g \cdot \mathcal{U}$ is also open in M, which means that $\pi(\mathcal{U})$ is open in M/G.

Definition 2.32 *The (topological) G-action on M is* proper *when the subsets $G_{A,B}$ are compact in G whenever A, B are compact subsets of M.*

This definition of proper action is equivalent to the condition that the map $\psi : G \times M \to M \times M$, $(g, m) \mapsto (g \cdot m, m)$ is *proper*, i.e. $\psi^{-1}(\text{compact}) = \text{compact}$. Note that the action of a compact group is always proper.

Proposition 2.33 *If a topological space M is equipped with a* proper *continuous action of a topological group G. The quotient topology is Hausdorff, locally compact.*

The proof is left to the reader. The main result is the following

Theorem 2.34 *Let M be a manifold equipped with a smooth, proper and free action of a Lie group. Then the quotient M/G equipped with the quotient topology carries the structure of a smooth manifold. Moreover the projection $\pi : M \to M/G$ is smooth, and any $n \in M/G$ has an open neighborhood $\mathcal{U}$ such that*

$$\begin{array}{ccc} \pi^{-1}(\mathcal{U}) & \xrightarrow{\sim} & \mathcal{U} \times G \\ m & \longmapsto & (\pi(m), \phi_{\mathcal{U}}(m)) \end{array}$$

is a G-equivariant diffeomorphism. Here $\phi_{\mathcal{U}} : \pi^{-1}(\mathcal{U}) \to G$ is an equivariant map : $\phi_{\mathcal{U}}(g \cdot m) = g\phi_{\mathcal{U}}(m)$.

[3]Here the topological spaces are Hausdorff and locally compact.

For a proof see [1][Section 2.3].

Remark 2.35 *Suppose that G is a discrete group. For a proper and free action of G on M we have: any $m \in M$ has an neighborhood $\mathcal{V}$ such that $g\mathcal{V} \cap \mathcal{V} = \emptyset$ for every $g \in G$, $g \neq e$. Theorem 2.34 is true when G is a discrete group. The quotient map $\pi : M \to M/G$ is then a covering map.*

The typical example we are interested in is the action of translation of a closed subgroup H of a Lie group G: the action of $h \in H$ is $G \to G, g \to gh^{-1}$. Its an easy exercise to see that this action is free and proper. The quotient space G/H is a smooth manifold and the action of translation $g \to ag$ of G on itself descend to a smooth action of G on G/H. The manifolds G/H are called homogeneous manifolds : these are the manifold with a transitive action of a Lie group G.

Stiefel manifolds, Grassmanians : Let V be a (real) vector space of dimension n. for any integer $k \leq n$, let $\mathrm{Hom}(\mathbb{R}^k, V)$ be the vector space of homomorphism equipped with the following (smooth) $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$-action: for $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$ and $f \in \mathrm{Hom}(\mathbb{R}^k, V)$, we take $(g, h) \cdot f(x) = g(f(h^{-1}x))$ for any $x \in \mathbb{R}^k$. Let $S_k(V)$ be the open subset of $\mathrm{Hom}(\mathbb{R}^k, V)$ formed by the one-to-one linear map : we have a natural identification of $S_k(V)$ with the set of families $\{v_1, \ldots, v_k\}$ of linearly independent vectors of V. Moreover $S_k(V)$ is stable under the $\mathrm{GL}(V) \times \mathrm{GL}(\mathbb{R}^k)$-action : the $\mathrm{GL}(V)$-action on $S_k(V)$ is *transitive*, and the $\mathrm{GL}(\mathbb{R}^k)$-action on $S_k(V)$ is *free and proper*. The manifold $S_k(V)/\mathrm{GL}(\mathbb{R}^k)$ admit a natural identification with the set $\{E \text{ subspace of } V \mid \dim E = k\}$: it is the grassmanian manifold $\mathrm{Gr}_k(V)$. On the other hand the action of $\mathrm{GL}(V)$ on $\mathrm{Gr}_k(V)$ is transitive so that

$$\mathrm{Gr}_k(V) \cong \mathrm{GL}(V)/H$$

where H is the closed Lie subgroup of $\mathrm{GL}(V)$ that fixes a subspace $E \subset V$ of dimension k.

2.9 Adjoint group

Let $\mathfrak{g}$ be a (real) Lie algebra. The automorphism group of $\mathfrak{g}$ is

$$\mathrm{Aut}(\mathfrak{g}) := \{\phi \in \mathrm{GL}(\mathfrak{g}) \,|\, \phi([X, Y]) = [\phi(X), \phi(Y)], \ \forall X, Y \in \mathfrak{g}\} \tag{2.20}$$

It is a closed subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra equal to

$$\mathrm{Der}(\mathfrak{g}) := \{D \in \mathfrak{gl}(\mathfrak{g}) \,|\, D([X, Y]) = [D(X), Y] + [X, D(Y)], \ \forall X, Y \in \mathfrak{g}\} \tag{2.21}$$

The subspace $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is called the set of *derivations* of $\mathfrak{g}$. Thanks to the Jacobi identity we know that $\mathrm{ad}(X) \in \mathrm{Der}(\mathfrak{g})$ for all $X \in \mathfrak{g}$. So the image of the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, that we denote $\mathrm{ad}(\mathfrak{g})$, is a Lie subalgebra of $\mathrm{Der}(\mathfrak{g})$.

Definition 2.36 *The adjoint group* $\mathrm{Ad}(\mathfrak{g})$ *is the connected Lie subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *associated to the Lie subalgebra of* $\mathrm{ad}(\mathfrak{g}) \subset \mathrm{Der}(\mathfrak{g})$*. As an abstract group, it is the subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *generated by the elements* $\mathrm{e}^{\mathrm{ad}(X)}$*,* $X \in \mathfrak{g}$*.*

Consider now a connected Lie group G, with Lie algebra $\mathfrak{g}$, and the adjoint map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. In this case, $\mathrm{e}^{\mathrm{ad}(X)} = \mathrm{Ad}(\exp_G(X))$ for any $X \in \mathfrak{g}$, so the image of G by Ad is equal to the group $\mathrm{Ad}(\mathfrak{g})$. If $g \in G$ belongs to the kernel of Ad, we have $g \exp_G(X) g^{-1} = \exp_G(\mathrm{Ad}(g)X) = \exp_G(X)$, so g commutes with all the element of $\exp_G(\mathfrak{g})$. But since G is connected, $\exp_G(\mathfrak{g})$ generates G. Finally we have proved that the kernel of Ad is equal to the center Z_G of the Lie group G.

It is worth to keep in mind the exact sequence of Lie group

$$0 \longrightarrow Z_G \longrightarrow G \longrightarrow \mathrm{Ad}(\mathfrak{g}) \longrightarrow 0 \qquad (2.22)$$

2.10 The Killing form

We have already defined the notions of solvable and nilpotent Lie algebra (see Def. 2.27). We have the following "opposite" notion.

Definition 2.37 *Let* $\mathfrak{g}$ *be (real) Lie algebra.*

• $\mathfrak{g}$ *is* simple *if* $\mathfrak{g}$ *is not abelian and does not contains ideals different from* $\{0\}$ *and* $\mathfrak{g}$*.*

• $\mathfrak{g}$ *is* semi-simple *if* $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ *where the* $\mathfrak{g}_i$ *are ideals of* $\mathfrak{g}$ *which are simple (as Lie algebras).*

The following remarks follows directly from the definition and give a first idea of the difference between "solvable" and "semi-simple".

Exercise 2.38 *Let* $\mathfrak{g}$ *be a (real) Lie algebra.*

• *Suppose that* $\mathfrak{g}$ *is solvable. Show that* $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$*, and that* $\mathfrak{g}$ *possess a non-zero abelian ideal.*

• *Suppose that* $\mathfrak{g}$ *is semi-simple. Show that* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$*, and show that* $\mathfrak{g}$ *does not possess non-zero abelian ideals : in particular the center* $Z_{\mathfrak{g}}$ *is reduced to* $\{0\}$*.*

In order to give the characterization of semi-simplicity we define the *Killing form* of a Lie algebra $\mathfrak{g}$. It is the symmetric $\mathbb{R}$-bilinear map $B_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ defined by

$$B_{\mathfrak{g}}(X, Y) = \mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)), \tag{2.23}$$

where $\mathrm{Tr} : \mathfrak{gl}(\mathfrak{g}) \to \mathbb{R}$ is the canonical trace map.

Proposition 2.39 *For $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $D \in \mathrm{Der}(\mathfrak{g})$ we have*
- *$B_{\mathfrak{g}}(\phi(X), \phi(Y)) = B_{\mathfrak{g}}(X, Y)$, and*
- *$B_{\mathfrak{g}}(DX, Y) + B_{\mathfrak{g}}(X, DY) = 0$ for all $X, Y \in \mathfrak{g}$.*
- *We have $B_{\mathfrak{g}}([X, Z], Y) = B_{\mathfrak{g}}(X, [Z, Y])$ for all $X, Y, Z \in \mathfrak{g}$.*

PROOF :If ϕ is an automorphism of $\mathfrak{g}$, we have $\mathrm{ad}(\phi(X)) = \phi \circ \mathrm{ad}(X) \circ \phi^{-1}$ for all $X \in \mathfrak{g}$ (see (2.20)). Then $a)$ follows and $b)$ comes from the derivative of $a)$ at $\phi = e$. For $c)$ take $D = \mathrm{ad}(Z)$ in $b)$. □

We recall now the basic interaction between the Killing form and the ideals of $\mathfrak{g}$. If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then
- the restriction of the Killing form of $\mathfrak{g}$ on $\mathfrak{h} \times \mathfrak{h}$ is the Killing form of $\mathfrak{h}$,
- the subspace $\mathfrak{h}^{\perp} = \{X \in \mathfrak{g} \mid B_{\mathfrak{g}}(X, \mathfrak{h}) = 0\}$ is an ideal of $\mathfrak{g}$.
- the intersection $\mathfrak{h} \cap \mathfrak{h}^{\perp}$is an ideal of $\mathfrak{g}$ with a Killing form identically equal to 0.

It was shown by E. Cartan that the Killing form gives criterion for semi-simplicity and solvability.

Theorem 2.40 (Cartan's Criterion for Semisimplicity) *Let $\mathfrak{g}$ be a (real) Lie algebra. The following statements are equivalent*
- *$\mathfrak{g}$ is semi-simple,*
- *the Killing form $B_{\mathfrak{g}}$ is non degenerate,*
- *$\mathfrak{g}$ does not have non-zero abelian ideals.*

The proof of Theorem 2.40 need the following characterization of the solvable Lie algebra. The reader will find a proof of the following theorem in [3][Section I].

Theorem 2.41 (Cartan's Criterion for Solvability) *Let $\mathfrak{g}$ be a (real) Lie algebra. The following statements are equivalent*
- *$\mathfrak{g}$ is solvable,*
- *$B_{\mathfrak{g}}(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

We will not prove Theorem 2.41, but only use the following easy corollary.

Corollary 2.42 *If $\mathfrak{g}$ is a (real) Lie algebra with $B_{\mathfrak{g}} = 0$, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.*

Before giving a proof of Theorem 2.40 let us show how Corollary 2.42 gives the implication $b) \Rightarrow a)$ in Theorem 2.41.

If $\mathfrak{g}$ is a Lie algebra with $B_{\mathfrak{g}} = 0$, then Corollary 2.42 tell us that $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$ different from $\mathfrak{g}$ with $B_{\mathfrak{g}^1} = 0$. If $\mathfrak{g}^1 \neq 0$, we do it again: $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}^1]$ is an ideal of $\mathfrak{g}^1$ different from $\mathfrak{g}^1$ with $B_{\mathfrak{g}^2} = 0$. This induction ends after finite steps: let $i \geq 0$ such that $\mathfrak{g}^i \neq 0$ and $\mathfrak{g}^{i+1} = 0$. Then $\mathfrak{g}^i$ is an abelian ideal of $\mathfrak{g}$, and $\mathfrak{g}$ is solvable. In the situation $b)$ of Theorem 2.41, we have then that $[\mathfrak{g}, \mathfrak{g}]$ is solvable, so $\mathfrak{g}$ is also solvable.

Proof of Theorem 2.40 using Corollary 2.42 :
$c) \Longrightarrow b)$. The ideal $\mathfrak{g}^{\perp} = \{X \in \mathfrak{g} \,|\, B_{\mathfrak{g}}(X, \mathfrak{g}) = 0\}$ of $\mathfrak{g}$ as a zero Killing form. If $\mathfrak{g}^{\perp} \neq 0$ we know from the preceding remark that there exists $i \geq 0$ such that $(\mathfrak{g}^{\perp})^i \neq 0$ and $(\mathfrak{g}^{\perp})^{i+1} = 0$. We see easily that $(\mathfrak{g}^{\perp})^i$ is also an ideal of $\mathfrak{g}$ (which is abelian). It gives a contradiction, then $\mathfrak{g}^{\perp} = 0$: the Killing form $B_{\mathfrak{g}}$ is non-degenerate.

$b) \Longrightarrow a)$. We suppose now that $B_{\mathfrak{g}}$ is non-degenerate. It gives first that $\mathfrak{g}$ is not abelian. After we use the following dichotomy:

$i)$ either $\mathfrak{g}$ does not have ideals different from $\{0\}$ and $\mathfrak{g}$, hence $\mathfrak{g}$ is simple,

$ii)$ either $\mathfrak{g}$ have an ideal $\mathfrak{h}$ different from $\{0\}$ and $\mathfrak{g}$.

In case $i)$ we have finish. In case $ii)$, let us show that $\mathfrak{h} \cap \mathfrak{h}^{\perp} \neq 0$: since $B_{\mathfrak{g}}$ is non-degenerate, it will implies that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. If $\mathfrak{a} := \mathfrak{h} \cap \mathfrak{h}^{\perp} \neq 0$, the Killing form on $\mathfrak{a}$ is equal to zero. Following Corollary 2.42 there exists $i \geq 0$ such that $\mathfrak{a}^i \neq 0$ and $\mathfrak{a}^{i+1} = 0$. Moreover since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, $\mathfrak{a}^i$ is also an ideal of $\mathfrak{g}$. By considering a supplementary F of $\mathfrak{a}^i$ in $\mathfrak{g}$, every endomorphism $\mathrm{ad}(X), X \in \mathfrak{g}$ as the following matricial expression

$$\mathrm{ad}(X) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix},$$

with $A : \mathfrak{a}^i \to \mathfrak{a}^i$, $B : F \to \mathfrak{a}^i$, and $D : F \to F$. The zero term is due to the fact that $\mathfrak{a}^i$ is an ideal of $\mathfrak{g}$. If $X_o \in \mathfrak{a}^i$, then

$$\mathrm{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}.$$

because $\mathfrak{a}^i$ is an abelian ideal. Finally for every $X \in \mathfrak{g}$,

$$\mathrm{ad}(X)\mathrm{ad}(X_o) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

and then $B_{\mathfrak{g}}(X, X_o) = 0$. It is a contradiction since $B_{\mathfrak{g}}$ is non-degenerate.

So if $\mathfrak{h}$ is an ideal different from $\{0\}$ and $\mathfrak{g}$, we have the $B_{\mathfrak{g}}$-orthogonal decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. Since $B_{\mathfrak{g}}$ is non-degenerate we see that $B_{\mathfrak{h}}$ and $B_{\mathfrak{h}^{\perp}}$ are non-degenerate, and we apply the dichotomy to the Lie algebras $\mathfrak{h}$ and $\mathfrak{h}^{\perp}$. After finite steps we obtain a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_k$ are simple ideals of $\mathfrak{g}$.

a) $\Longrightarrow$ *c)*. Let $p_k : \mathfrak{g} \to \mathfrak{g}_k$ be the projections relative to a decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$ in simple ideals: the p_k are Lie algebras morphims. If $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, each $p_k(\mathfrak{a})$ is an abelian ideal of $\mathfrak{g}_k$ which is equal to $\{0\}$ since $\mathfrak{g}_k$ is simple. It proves that $\mathfrak{a} = 0$. $\square$

Exercise 2.43 • *For the Lie algebra* $\mathfrak{sl}(n, \mathbb{R})$ *show that* $B_{\mathfrak{sl}(n,\mathbb{R})}(X, Y) = 2n\mathrm{Tr}(XY)$. *Conclude that* $\mathfrak{sl}(n, \mathbb{R})$ *is a semi-simple Lie algebra.*

• *For the Lie algebra* $\mathfrak{su}(n)$ *show that* $B_{\mathfrak{su}(n)}(X, Y) = 2n\mathrm{Re}(\mathrm{Tr}(XY))$. *Conclude that* $\mathfrak{su}(n)$ *is a semi-simple Lie algebra.*

Exercise 2.44 $\mathfrak{sl}(n, \mathbb{R})$ *is a simple Lie algebra.*

Let $(E_{i,j})_{1\leq i,j\leq n}$ *be the canonical basis of* $\mathfrak{gl}(\mathbb{R}^n)$. *Consider a non-zero ideal* $\mathfrak{a}$ *of* $\mathfrak{sl}(n, \mathbb{R})$. *Up to a change of* $\mathfrak{a}$ *in* $\mathfrak{a}^{\perp}$ *we can assume that* $\dim(\mathfrak{a}) \geq \frac{n^2-1}{2}$.

• *Show that* $\mathfrak{a}$ *possess an element* X *which is not diagonal.*

• *Compute* $[[X, E_{i,j}], E_{i,j}]$ *and conclude that some* $E_{i,j}$ *with* $i \neq j$ *belongs to* $\mathfrak{a}$.

• *Show that* $E_{k,l}, E_{k,k} - E_{l,l} \in \mathfrak{a}$ *when* $k \neq l$. *Conclude.*

2.11 Complex Lie algebras

We worked out the notions of solvable, nilpotent, simple and semi-simple *real* Lie algebras. The definitions go through for Lie algebras defined over any field k, and all the result of section 2.10 are true for $k = \mathbb{C}$.

Let $\mathfrak{h}$ be a *complex* Lie algebra. The Killing form is here a symmetric $\mathbb{C}$-bilinear map $B_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ defined by (2.23), where $\mathrm{Tr} : \mathfrak{gl}_{\mathbb{C}}(\mathfrak{h}) \to \mathbb{C}$ is the trace defined on the $\mathbb{C}$-linear endomorphism of $\mathfrak{h}$.

Theorem 2.40 is valid for the complex Lie algebras: a *complex* Lie algebra is direct sum of simple ideals if and only if its Killing form is non-degenerate.

A usefull toll is the complexification of *real* Lie algebras. If $\mathfrak{g}$ is a real Lie algebra, the complexified vector space $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ carries a canonical structure of complex Lie algebras. We see easily that the Killing forms $B_{\mathfrak{g}}$ and $B_{\mathfrak{g}_{\mathbb{C}}}$ coincide on $\mathfrak{g}$:

$$B_{\mathfrak{g}_{\mathbb{C}}}(X, Y) = B_{\mathfrak{g}}(X, Y) \quad \text{for all } X, Y \in \mathfrak{g}. \tag{2.24}$$

With (2.24) we see that a real Lie algebra $\mathfrak{g}$ is semi-simple if and only if the complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is semi-simple.

3 Semi-simple Lie groups

Definition 3.1 *A connected Lie group G is* semi-simple *(resp.* simple*) if its Lie algebra $\mathfrak{g}$ is semi-simple (resp.* simple*).*

If we use Theorem 2.40 and Proposition 2.25 we have the following equivalent characterization of semi-simple Lie group that will be used in the lecture of J. Maubon (see Proposition 6.3).

Proposition 3.2 *A connected Lie group G is* semi-simple *if and only if G does not have non-trivial connected normal abelian Lie subroup.*

In particular the center Z_G of a semi-simple Lie group is discrete. We have the following refinement for the simple Lie groups.

Proposition 3.3 *A normal subgroup A of a (connected) simple Lie Group G which is not equal to G belongs to the center Z of G.*

Proof : Let A_o be subset of A defined as follow : $a \in A_o$ if there exits a continuous curve $c(t)$ in A with $c(0) = e$ and $c(1) = a$. Obviously A_o is a path-connected subgroup of G, so according to Theorem 2.23 A_o is a Lie subgroup of G. If $c(t)$ is continuous curve in A, $gc(t)g^{-1}$ is also a continuous curve in A for all $g \in G$, and then A_o is a normal subgroup of G. From Proposition 2.25 we know that the Lie algebra of A_o is an ideal of $\mathfrak{g}$, hence is equal to $\{0\}$ since $\mathfrak{g}$ is simple and $A \neq G$. We have proved that $A_o = \{e\}$, which means that every continuous curve in A is constant. For every $a \in A$ and all continous curve $\gamma(t)$ in G, the continuous curve $\gamma(t)a\gamma(t)^{-1}$ in A must be constant. It proves that A belongs to the center of G. □

We come back to the exact sequence (2.22).

Lemma 3.4 *If $\mathfrak{g}$ is a semi-simple Lie algebra, the vector space of derivation* $\mathrm{Der}(\mathfrak{g})$ *is equal to* $\mathrm{ad}(\mathfrak{g})$.

Proof : Let D be a derivation of $\mathfrak{g}$. Since $B_{\mathfrak{g}}$ is non-degenerate there exist a unique $X_D \in \mathfrak{g}$ such that $\mathrm{Tr}(D\mathrm{ad}(Y)) = B_{\mathfrak{g}}(X_D, Y)$, for all $Y \in \mathfrak{g}$.

Now we compute

$$\begin{aligned} B_{\mathfrak{g}}([X_D, Y], Z]) = B_{\mathfrak{g}}(X_D, [Y, Z]) &= \mathrm{Tr}(D\mathrm{ad}([Y, Z])) \\ &= \mathrm{Tr}(D[\mathrm{ad}(Y), \mathrm{ad}(Z)]) \\ &= \mathrm{Tr}([D, \mathrm{ad}(Y)]\mathrm{ad}(Z)) \qquad (1) \\ &= \mathrm{Tr}(\mathrm{ad}(DY)\mathrm{ad}(Z)) \qquad (2) \\ &= B_{\mathfrak{g}}(DY, Z). \end{aligned}$$

(1) is a general fact about the trace: $\mathrm{Tr}(A[B, C]) = \mathrm{Tr}([A, B]C)$ for any $A, B, C \in \mathfrak{gl}(\mathfrak{g})$. (2) uses the definition of a derivation (see (2.21)). Using now the non-degeneracy of $B_{\mathfrak{g}}$ we get $D = \mathrm{ad}(X_D)$. □

The equality of Lie algebras $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$ tells us that the adjoint group is equal to identity component of the automorphism group: $\mathrm{Ad}(\mathfrak{g}) = \mathrm{Aut}(\mathfrak{g})_o$.

Lemma 3.5 *If G is a (connected) semi-simple Lie group, it's center Z_G is discrete and the adjoint group as zero center.*

PROOF :The center $Z(G)$ is discrete because the semi-simple Lie algebra $\mathfrak{g}$ as zero center. Let $\mathrm{Ad}(g)$ be an element of the center of $\mathrm{Ad}(\mathfrak{g})$: we have

$$\begin{aligned} \mathrm{Ad}(\exp_G(X)) = \mathrm{Ad}(g)\mathrm{Ad}(\exp_G(X))\mathrm{Ad}(g)^{-1} &= \mathrm{Ad}(g \exp_G(X) g^{-1}) \\ &= \mathrm{Ad}(\exp_G(\mathrm{Ad}(g)X)) \end{aligned}$$

for any $X \in \mathfrak{g}$. So $\exp_G(-X)\exp_G(\mathrm{Ad}(g)X) \in Z(G)$, $\forall X \in \mathfrak{g}$. But since $Z(G)$ is discrete it implies that $\exp_G(X) = \exp_G(\mathrm{Ad}(g)X))$, $\forall X \in \mathfrak{g}$: g commutes with any element of $\exp_G(\mathfrak{g})$. Since $\exp_G(\mathfrak{g})$ generates G, we have finally that $g \in Z(G)$ and so $\mathrm{Ad}(g) = 1$. □

The important point here is that a (connected) semi-simple Lie group is a central extension by a discrete subgroup of a quasi-algebraic group. The Lie group $\mathrm{Aut}(\mathfrak{g})$ is defined by finite polynomial identities in $\mathrm{GL}(\mathfrak{g})$: it is an algebraic group. And $\mathrm{Ad}(\mathfrak{g})$ is a connected component of $\mathrm{Aut}(\mathfrak{g})$: it is a quasi-algebraic group. There is an important case where the Lie algebra structure impose some restriction on the center.

Theorem 3.6 (Weyl) *Let G be a connected Lie group such that $B_{\mathfrak{g}}$ is* negative definite. *Then G is a* compact *semi-simple Lie group and the center Z_G is finite.*

There are many proofs, for example [2][Section II.6], [1][Section 3.9]. Here we only stress that the condition "$B_{\mathfrak{g}}$ is negative definite" imposes that $\mathrm{Aut}(\mathfrak{g})$ is a compact subgroup of $\mathrm{GL}(\mathfrak{g})$, hence $\mathrm{Ad}(\mathfrak{g})$ is compact. Now if we consider the exact sequence $0 \to Z_G \to G \to \mathrm{Ad}(\mathfrak{g}) \to 0$ we see that G is compact if and only if Z_G is finite.

Definition 3.7 *A real Lie algebra is* compact *if its Killing form is negative definite.*

3.1 Cartan decomposition on subgroups of $\mathrm{GL}(\mathbb{R}^n)$

Let Sym_n be the vector subspace of $\mathfrak{gl}(\mathbb{R}^n)$ formed by the symmetric endomorphisms, and let Sym_n^+ be the open subspace of Sym_n formed by the positive definite symmetric endomorphisms. Consider the exponential $\mathrm{e} : \mathfrak{gl}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$. We compute its differential.

Lemma 3.8 *For any $X \in \mathfrak{gl}(\mathbb{R}^n)$, the tangent map $\mathbf{T}_X\mathrm{e} : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ is equal to $\mathrm{e}^X \left(\frac{1-\mathrm{e}^{-\mathrm{ad}(X)}}{\mathrm{ad}(X)}\right)$. In particular, $\mathbf{T}_X\mathrm{e}$ is a singular map if and only if the adjoint map $\mathrm{ad}(X) : \mathfrak{gl}(\mathbb{R}^n) \to \mathfrak{gl}(\mathbb{R}^n)$ has a non-zero eigenvalue belonging to $2i\pi\mathbb{Z}$.*

Proof : Consider the smooth functions $F(s,t) = \mathrm{e}^{s(X+tY)}$, and $f(s) = \frac{\partial F}{\partial t}(s,0)$: we have $f(0) = 0$ and $f(1) = \mathbf{T}_X\mathrm{e}(Y)$. If we differentiate F first with respect to t, and after with respect to s, we find that f satisfies the differential equation $f'(s) = Y\mathrm{e}^{sX} + Xf(s)$ which equivalent to

$$(\mathrm{e}^{-sX}f)' = \mathrm{e}^{-sX}Y\mathrm{e}^{-sX} = \mathrm{e}^{-s\,\mathrm{ad}(X)}Y.$$

Finally we find $f(1) = \mathrm{e}^X(\int_0^1 \mathrm{e}^{-s\,\mathrm{ad}(X)}ds)Y$. $\Box$

It is easy exercise to show that exponential map realize a one-to-one map from Sym_n onto Sym_n^+. The last Lemma tells us that $\mathbf{T}_X\mathrm{e}$ is not singular for every $X \in \mathrm{Sym}_n$. So we have prove the

Lemma 3.9 *The exponential map $A \mapsto \mathrm{e}^A$ realizes a smooth diffeomorphism from Sym_n onto Sym_n^+.*

Let $\mathrm{O}(\mathbb{R}^n)$ the orthogonal group : $k \in \mathrm{O}(\mathbb{R}^n) \iff {}^t kk = Id$. Every $g \in \mathrm{GL}(\mathbb{R}^n)$ decomposes in a unique manner as $g = kp$ where $k \in \mathrm{O}(\mathbb{R}^n)$ and $p \in \mathrm{Sym}_n^+$ is the square root of ${}^t gg$. The map $(k,p) \mapsto kp$ defines a smooth diffeomorphism from $\mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n^+$ onto $\mathrm{GL}(\mathbb{R}^n)$. If we use Lemma 3.9, we get the following

Proposition 3.10 (Cartan decomposition) *The map*

$$\begin{array}{rcl} \mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n & \longrightarrow & \mathrm{GL}(\mathbb{R}^n) \\ (k, X) & \longmapsto & k\mathrm{e}^X \end{array} \tag{3.25}$$

is a smooth diffeomorphism.

We will now extend the Cartan decomposition to an algebraic[4] subgroup G of $\mathrm{GL}(\mathbb{R}^n)$ which is stable under the *transpose map*. In other term G is stable under the automorphism $\Theta_o : \mathrm{GL}(\mathbb{R}^n) \to \mathrm{GL}(\mathbb{R}^n)$ defined by

$$\Theta_o(g) = {}^t g^{-1}. \tag{3.26}$$

The classical groups like $\mathrm{SL}(n, \mathbb{R})$, $\mathrm{O}(p, q)$, $\mathrm{Sp}(\mathbb{R}^{2n})$ fall into this category. The Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ of G is stable under the transpose map, so we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{o}(n, \mathbb{R})$ and $\mathfrak{p} = \mathfrak{g} \cap \mathrm{Sym}_n$.

Lemma 3.11 *Let $X \in \mathrm{Sym}_n$ such that $\mathrm{e}^X \in G$. Then $\mathrm{e}^{tX} \in G$ for every $t \in \mathbb{R}$: in other word $X \in \mathfrak{p}$.*

PROOF : The element e^X can be diagonalized : there exist $g \in \mathrm{GL}(\mathbb{R}^n)$ and a sequence of real number $\lambda_1 \ldots \lambda_n$ such that $\mathrm{e}^{tX} = g\,\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})g^{-1}$ for all $t \in \mathbb{R}$ (here $\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})$ is a diagonal matrix). From the hypothesis we have that $\mathrm{Diag}(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n})$ belongs to the algebraic group $g^{-1}Gg$ when $t \in \mathbb{Z}$. Now it an easy fact that for any polynomial in n-variables P, if $\phi(t) = P(\mathrm{e}^{t\lambda_1}, \ldots, \mathrm{e}^{t\lambda_n}) = 0$ for all $t \in \mathbb{Z}$, then ϕ is identically equal to 0. So we have prove that $\mathrm{e}^{tX} \in G$ for every $t \in \mathbb{R}$ whenever $\mathrm{e}^X \in G$. □

Consider the Cartan decomposition $g = k\mathrm{e}^X$ of an element $g \in G$. Since G is stable under the transpose map $\mathrm{e}^{2X} = {}^t gg \in G$. ¿From Lemma 3.11 we get that $X \in \mathfrak{p}$ and $k \in G \cap \mathrm{O}(\mathbb{R}^n)$. Finally, if we restrict the diffeomorphism 3.25 to the submanifold $(G \cap \mathrm{O}(\mathbb{R}^n)) \times \mathfrak{p} \subset \mathrm{O}(\mathbb{R}^n) \times \mathrm{Sym}_n$ we get a diffeomorphism

$$(G \cap \mathrm{O}(\mathbb{R}^n)) \times \mathfrak{p} \xrightarrow{\sim} G. \tag{3.27}$$

Let K be the connected Lie subgroup of G associated to the subalgebra $\mathfrak{k}$: K is equal to the identity component of the compact Lie group $G \cap \mathrm{O}(\mathbb{R}^n)$ hence K is compact. If we restrict the diffeomorphism (3.27) to the identity component G_o of G we get the diffeomorphism

$$K \times \mathfrak{p} \xrightarrow{\sim} G_o. \tag{3.28}$$

3.2 Cartan involutions

We start again with the situation of a closed subgroup G of $\mathrm{GL}(\mathbb{R}^n)$ stable under the transpose map $A \mapsto {}^t A$. Then the lie algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ of G is also stable under the transpose map.

[4]i.e. defined by a finite number of polynomial equalities.

Proposition 3.12 *If the Lie algebra* $\mathfrak{g}$ *as a center reduced to* 0*, then* $\mathfrak{g}$ *is semi-simple. In particular, the bilinear map* $(X, Y) \mapsto B_{\mathfrak{g}}(X, {}^tY)$ *defines a scalar product on* $\mathfrak{g}$*. Moreover if we consider the transpose map* $D \mapsto {}^{\mathrm{t}}D$ *on* $\mathfrak{gl}(\mathfrak{g})$ *defined by this scalar product, we have* $\mathrm{ad}({}^tX) = {}^{\mathrm{t}}\mathrm{ad}(X)$ *for all* $X \in \mathfrak{g}$*.*

PROOF : Consider the scalar product on $\mathfrak{g}$ defined by $(X, Y)_{\mathfrak{g}} := \mathrm{Tr}({}^tXY)$ where Tr is the canonical trace on $\mathfrak{gl}(\mathbb{R}^n)$. With the help of $(-,-)_{\mathfrak{g}}$, we have a transpose map $D \mapsto {}^{\mathrm{T}}D$ on $\mathfrak{gl}(\mathfrak{g})$: $(D(X), Y)_{\mathfrak{g}} = (X, {}^{\mathrm{T}}D(Y))_{\mathfrak{g}}$ for all $X, Y \in \mathfrak{g}$ and $D \in \mathfrak{gl}(\mathfrak{g})$. A small computation shows that ${}^{\mathrm{T}}\mathrm{ad}(X) = \mathrm{ad}({}^tX)$, and then $B_{\mathfrak{g}}(X, {}^tY) = \mathrm{Tr}'(\mathrm{ad}(X)\,{}^{\mathrm{T}}\mathrm{ad}(Y))$ is a symmetric bilinear map on $\mathfrak{g}\times\mathfrak{g}$ (here Tr' is the trace map on $\mathfrak{gl}(\mathfrak{g})$). If $\mathfrak{g}$ as zero center then $B_{\mathfrak{g}}(X, {}^tX) > 0$ if $X \neq 0$. Let $D \mapsto {}^{\mathrm{t}}D$ be the transpose map on $\mathfrak{gl}(\mathfrak{g})$ defined by this scalar product. We have

$$B_{\mathfrak{g}}(\mathrm{ad}(X)Y, {}^tZ) = -B_{\mathfrak{g}}(Y, [X, {}^tZ]) = B_{\mathfrak{g}}(Y, {}^t[{}^tX, Z]),$$

for all $X, Y, Z \in \mathfrak{g}$: in other terms $\mathrm{ad}({}^tX) = {}^{\mathrm{t}}\mathrm{ad}(X)$. □

Definition 3.13 *A linear map* $\tau : \mathfrak{g} \to \mathfrak{g}$ *on a Lie algebra is an* involution *if* τ *is an* automorphism *of the Lie algebra* $\mathfrak{g}$ *and* $\tau^2 = 1$*.*

When τ is an involution of $\mathfrak{g}$, we define the bilinear map

$$B^\tau(X, Y) := -B_{\mathfrak{g}}(X, \tau(Y)) \tag{3.29}$$

which is *symmetric*. We have the decomposition

$$\mathfrak{g} = \mathfrak{g}_1^\tau \oplus \mathfrak{g}_{-1}^\tau \tag{3.30}$$

where $\mathfrak{g}_{\pm 1}^\tau = \{X \in \mathfrak{g} \mid \tau(X) = \pm X\}$. Since $\tau \in \mathrm{Aut}(\mathfrak{g})$ we have

$$[\mathfrak{g}_\varepsilon^\tau, \mathfrak{g}_{\varepsilon'}^\tau] \subset \mathfrak{g}_{\varepsilon\varepsilon'}^\tau \quad \text{for all} \quad \varepsilon, \varepsilon' \in \{1, -1\}, \tag{3.31}$$

and

$$B_{\mathfrak{g}}(X, Y) = 0 \quad \text{for all} \quad X \in \mathfrak{g}_1^\tau,\ Y \in \mathfrak{g}_{-1}^\tau. \tag{3.32}$$

The subspace[5] $\mathfrak{g}^\tau$ is a sub-algebra of $\mathfrak{g}$, $\mathfrak{g}_{-1}^\tau$ is a module for $\mathfrak{g}^\tau$ through the adjoint action, and the subspace $\mathfrak{g}^\tau$ and $\mathfrak{g}_{-1}^\tau$ are *orthogonal* with respect to B^τ.

Definition 3.14 *An involution* θ *on a Lie algebra* $\mathfrak{g}$ *is a* Cartan involution *if the symmetric bilinear map* B^θ *defines a scalar product on* $\mathfrak{g}$*.*

[5] We will just denote by $\mathfrak{g}^\tau$ the subalgebra $\mathfrak{g}_1^\tau$.

Note that the existence of a Cartan involution implies the semi-simplicity of the Lie algebra.

EXAMPLE : $\theta_o(X) = -{}^tX$ is an involution on the Lie algebra $\mathfrak{gl}(\mathbb{R}^n)$. We prove in Proposition 3.12 that if a Lie sub-algebra $\mathfrak{g} \subset \mathfrak{gl}(\mathbb{R}^n)$ is stable under the transpose map and has zero center, then the linear θ_o restricted to $\mathfrak{g}$ is a Cartan involution. It is the case, for example, of the subalgebras $\mathfrak{sl}(n, \mathbb{R})$ and $\mathfrak{o}(p, q)$.

In the other direction, if a semi-simple Lie algebra $\mathfrak{g}$ is equipped with a Cartan involution θ, a small computation shows that

$$ {}^t\mathrm{ad}(X) = -\mathrm{ad}(\theta(X)), \quad X \in \mathfrak{g}, $$

where $A \mapsto {}^tA$ is the transpose map on $\mathfrak{gl}(\mathfrak{g})$ defined by the scalar product B^θ. So the subalgebra $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, which is isomorphic to $\mathfrak{g}$, is stable under the transpose map. Conclusion : for a real Lie algebra $\mathfrak{g}$ with zero center, the following statements are equivalent :

- $\mathfrak{g}$ can be realized as a subalgebra of matrices stable under the transpose map,
- $\mathfrak{g}$ is a semi-simple Lie algebra equipped with a Cartan involution.

In the next section, we will see that any real semi-simple Lie algebra has a Cartan involution.

3.3 Compact real forms

We have seen the notion of *complexification* of a real Lie algebra. In the other direction, a complex Lie algebra $\mathfrak{h}$ can be consider as a *real* Lie algebra and we denote it by $\mathfrak{h}^{\mathbb{R}}$. The behavior of the Killing form with respect to this operation is

$$ B_{\mathfrak{h}^{\mathbb{R}}}(X, Y) = 2\,\mathrm{Re}(B_{\mathfrak{h}}(X, Y)) \quad \text{for all } X, Y \in \mathfrak{h}. \tag{3.33} $$

For a complex Lie algebra $\mathfrak{h}$, we speak of *anti-linear involutions* : it is the involutions of $\mathfrak{h}^{\mathbb{R}}$ which anti-commute with the complex multiplication. If τ is an anti-linear involution of $\mathfrak{h}$ then $\mathfrak{h}^\tau_{-1} = i\mathfrak{h}^\tau$, i.e.

$$ \mathfrak{h} = \mathfrak{h}^\tau \oplus i\mathfrak{h}^\tau. \tag{3.34} $$

Definition 3.15 *A* real form *of a complex Lie algebra* $\mathfrak{h}$ *is a real subalgebra* $\mathfrak{a} \subset \mathfrak{h}^{\mathbb{R}}$ *such that* $\mathfrak{h} = \mathfrak{a} \oplus i\mathfrak{a}$*, i.e.* $\mathfrak{a}_{\mathbb{C}} \simeq \mathfrak{h}$*. A* compact real form *of a complex Lie algebra is a* real form *which is a* compact *Lie algebra (see Def. 3.7).*

For any real form $\mathfrak{a}$ of $\mathfrak{h}$, there exist a unique anti-linear involution τ such that $\mathfrak{h}^\tau = \mathfrak{a}$. Equation (3.34) tells us that $\tau \mapsto \mathfrak{h}^\tau$ is a one-to-one correspondence between the *anti-linear involutions* of $\mathfrak{h}$ and the *real forms* of $\mathfrak{h}$. If $\mathfrak{a}$ is a real form of a complex Lie algebra $\mathfrak{h}$, we have like in (2.24) that

$$B_{\mathfrak{a}}(X,Y) = B_{\mathfrak{h}}(X,Y) \quad \text{for all } X, Y \in \mathfrak{a} \tag{3.35}$$

In particular $B_{\mathfrak{h}}$ take real values on $\mathfrak{a} \times \mathfrak{a}$.

Lemma 3.16 *Let θ an anti-linear involution of a complex Lie algebra $\mathfrak{h}$. θ is a Cartan involution of the real Lie algebra $\mathfrak{h}^{\mathbb{R}}$ if and only if $\mathfrak{h}^\theta$ is a compact real form of $\mathfrak{h}$.*

PROOF : Consider the decomposition $\mathfrak{h} = \mathfrak{h}^\theta \oplus i\mathfrak{h}^\theta$ and $X = a + ib$ with $a, b \in \mathfrak{h}^\theta$. We have

$$\begin{aligned} B_{\mathfrak{h}^{\mathbb{R}}}(X, \theta(X)) &= 2(B_{\mathfrak{h}}(a,a) + B_{\mathfrak{h}}(b,b)) \qquad (1) \\ &= 2(B_{\mathfrak{h}^\theta}(a,a) + B_{\mathfrak{h}^\theta}(b,b)) \qquad (2). \end{aligned}$$

(1) and (2) are consequence of (3.33) and (3.35). So we see that $-B^\theta_{\mathfrak{h}^{\mathbb{R}}}$ is positive definite on $\mathfrak{h}^{\mathbb{R}}$ if and only if the Killing form $B_{\mathfrak{h}^\theta}$ is negative definite. □

EXAMPLE : the Lie algebra $\mathrm{sl}(n,\mathbb{R})$ is a real form of $\mathrm{sl}(n,\mathbb{C})$. The complex Lie algebra $\mathrm{sl}(n,\mathbb{C})$ as other real forms like

- $\mathrm{su}(n) = \{X \in \mathrm{sl}(n,\mathbb{C}) \mid {}^t\overline{X} + X = 0\}$,
- $\mathrm{su}(p,q) = \{X \in \mathrm{sl}(n,\mathbb{C}) \mid {}^t\overline{X}\mathrm{I}_{p,q} + \mathrm{I}_{p,q}X = 0\}$, where $\mathrm{I}_{p,q} = \begin{pmatrix} \mathrm{Id}_p & 0 \\ 0 & -\mathrm{Id}_q \end{pmatrix}$.

Here the anti-linear involutions are respectively $\sigma(X) = \overline{X}$, $\sigma_a(X) = -{}^t\overline{X}$, and $\sigma_b(X) = -\mathrm{I}_{p,q}{}^t\overline{X}\mathrm{I}_{p,q}$. Among the real forms $\mathrm{sl}(n,\mathbb{R})$, $\mathrm{su}(n)$, $\mathrm{su}(p,q)$ of $\mathrm{sl}(n,\mathbb{C})$, $\mathrm{su}(n)$ is the only one which is compact.

Let $\mathfrak{g}$ be a real Lie algebra, and let σ be the anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ associated to the real form $\mathfrak{g}$. We have a one-to-one correspondence

$$\tau \mapsto \mathfrak{u}(\tau) := (\mathfrak{g}_{\mathbb{C}})^{\tau\circ\sigma} \tag{3.36}$$

between the set of involution of $\mathfrak{g}$ and the set of real forms of $\mathfrak{g}_{\mathbb{C}}$ which are σ-stable. If τ is an involution of $\mathfrak{g}$, we consider its $\mathbb{C}$-linear extension to $\mathfrak{g}_{\mathbb{C}}$ (that we still denote by τ). The composite $\tau \circ \sigma = \sigma \circ \tau$ is then an anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ which commutes with σ: hence the real form $\mathfrak{u}(\tau) := (\mathfrak{g}_{\mathbb{C}})^{\tau\circ\sigma}$ is stable under σ. If $\mathfrak{a}$ is a real form on $\mathfrak{g}_{\mathbb{C}}$ defined by a anti-linear involution ρ which commutes with σ, then $\sigma \circ \rho$ is a $\mathbb{C}$-linear involution on $\mathfrak{g}_{\mathbb{C}}$ which commutes with σ: then it is the complexification of an involution τ on $\mathfrak{g}$, and we have $\mathfrak{a} = \mathfrak{u}(\tau)$.

Proposition 3.17 *Let $\mathfrak{g}$ be a real semi-simple Lie algebra. Let τ be an involution of $\mathfrak{g}$ and let $\mathfrak{u}(\tau)$ be the real form of $\mathfrak{g}_{\mathbb{C}}$ defined by (3.36). The following statements are equivalents*

- *τ is a Cartan involution of $\mathfrak{g}$,*
- *$\mathfrak{u}(\tau)$ is compact real form of $\mathfrak{g}_{\mathbb{C}}$ (which is σ-stable).*

PROOF : If $\mathfrak{g} = \mathfrak{g}^\tau \oplus \mathfrak{g}^\tau_{-1}$ is the decomposition related to the eigen-spaces of τ then $\mathfrak{u}(\tau) = \mathfrak{g}^\tau \oplus i\, \mathfrak{g}^\tau_{-1}$. Take $X = a + ib \in \mathfrak{u}(\tau)$ with $a \in \mathfrak{g}^\tau$ and $b \in \mathfrak{g}^\tau_{-1}$. We have

$$\begin{aligned} B_{\mathfrak{u}(\tau)}(X,X) &= B_{\mathfrak{g}_{\mathbb{C}}}(X,X) \qquad (1) \\ &= B_{\mathfrak{g}}(a,a) - B_{\mathfrak{g}}(b,b) \quad (2) \\ &= -B^\tau_{\mathfrak{g}}(\tilde{X},\tilde{X}), \end{aligned}$$

where $\tilde{X} = a + b \in \mathfrak{g}$. (1) is due to (3.35). In (2) we use (2.24) and the fact that $\mathfrak{g}^\tau$ and $\mathfrak{g}^\tau_{-1}$ are $B_{\mathfrak{g}}$-orthogonal. Then we see that $B_{\mathfrak{u}(\tau)}$ is negative definite if and only if $B^\tau_{\mathfrak{g}}$ is positive definite. □

Now we give the way we can prove that a real semi-simple Lie algebra $\mathfrak{g}$ has a Cartan involution. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of $\mathfrak{g}$ and let σ the anti-linear involution of $\mathfrak{g}_{\mathbb{C}}$ corresponding to the real form $\mathfrak{g}$. We now from Proposition 3.17 that it is equivalent to look to the σ-stable compact real forms of $\mathfrak{g}_{\mathbb{C}}$. We use first the following fundamental fact.

Theorem 3.18 *Any complex semi-simple Lie algebra has a compact real form.*

A proof can be found in [3][Section 7.1]. The existence of a σ-stable compact real form is given by the following

Lemma 3.19 *Let $\tau : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ be anti-linear involution corresponding to a compact real form of $\mathfrak{g}_{\mathbb{C}}$. There exists $\phi \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ such that the anti-linear involution $\phi\tau\phi^{-1}$ commutes with σ. Hence $\phi\tau\phi^{-1}|_{\mathfrak{g}}$ is a Cartan involution of to $\mathfrak{g}$.*

PROOF : The complex vector space $\mathfrak{g}_{\mathbb{C}}$ is equipped with the hermitian metric : $(X,Y) \to B_{\mathfrak{g}_{\mathbb{C}}}(X, \tau(Y))$. It easy to check that $\tau\sigma$ belongs to the intersection

$$\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \cap \{\text{hermitian endomorphism}\} = \{\phi \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}) \ \mid \ \tau\phi\tau = \phi^{-1}\} \tag{3.37}$$

$\rho = (\tau\sigma)^2$ is positive definite. Following Lemma 3.11, the one parameter subgroup $r \in \mathbb{R} \mapsto \rho^r$ belongs to the identity component $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})_o$ (since $\mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ is an algebraic subgroup of $\mathrm{GL}((\mathfrak{g}_{\mathbb{C}})^{\mathbb{R}})$). We leave as an exercise to check that ρ^r commutes with $\tau\sigma$ for all $r \in \mathbb{R}$. Since $\tau\rho^r\tau = \rho^{-r}$ (see (3.37)) it is easy to see that $\rho^r\tau\rho^{-r}$ commutes with σ if $r = \frac{-1}{4}$. □

3.4 Cartan decomposition at the group level

Let G be a connected semi-simple Lie group with Lie algebra $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$. So we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{p} = \mathfrak{g}^\theta_{-1}$ is a $\mathfrak{k}$-module. Let K be the connected Lie subgroup of G associated to $\mathfrak{k}$. This section is devoted to the proof of the following

Theorem 3.20 *(a) K is a closed subgroup of G*

(b) the mapping $K \times \mathfrak{p} \to G$ given by $(k, X) \mapsto k \exp_G(X)$ is a diffeomorphism onto

(c) K contains the center Z of G

(d) K is compact if and only if Z is finite

(e) there exists a Lie group automorphism Θ of G, with $\Theta^2 = 1$ and with differential θ

(f) the subgroup of G fixed by Θ is K.

Proof : The Lie group $\widehat{G} = \mathrm{Ad}(\mathfrak{g})$ which is equal to the image of G by the adjoint action is the identity component of $\mathrm{Aut}(\mathfrak{g})$. The Lie algebra $\widehat{\mathfrak{g}}$ of $\widehat{G}$ which is equal to the subspace of derivations $\mathrm{Der}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is stable under the transpose map $A \mapsto {}^tA$ on $\mathfrak{gl}(\mathfrak{g})$ associated to the scalar product B_θ on $\mathfrak{g}$ (since $-{}^t\mathrm{ad}(X) = \mathrm{ad}(\theta(X))$). Since $\widehat{G}$ is generated by $\mathrm{e}^{\mathrm{ad}(X)}$, $X \in \mathfrak{g}$, $\widehat{G}$ is stable under the group morphism $A \mapsto {}^tA^{-1}$. We have $\widehat{\mathfrak{g}} = \widehat{\mathfrak{k}} \oplus \widehat{\mathfrak{p}}$ where $\widehat{\mathfrak{k}} = \{A \in \widehat{\mathfrak{g}} \mid {}^tA = -A\}$ and $\widehat{\mathfrak{p}} = \{A \in \widehat{\mathfrak{g}} \mid {}^tA = A\}$. We have of course $\widehat{\mathfrak{g}} = \mathrm{ad}(\mathfrak{g})$, $\widehat{\mathfrak{k}} = \mathrm{ad}(\mathfrak{k})$ and $\widehat{\mathfrak{p}} = \mathrm{ad}(\mathfrak{p})$. Let $\widehat{K}$ be the compact Lie group equal to $\widehat{G} \cap \mathrm{O}(\mathfrak{g})$: its Lie algebra is $\widehat{\mathfrak{k}}$. Since $\mathrm{Aut}(\mathfrak{g})$ is an algebraic subgroup of $\mathrm{GL}(\mathfrak{g})$, (3.28) applies here and gives the diffeomorphism

$$\begin{array}{ccc} \widehat{K} \times \widehat{\mathfrak{p}} & \longrightarrow & \widehat{G} \\ (k, A) & \longmapsto & k\mathrm{e}^A. \end{array} \tag{3.38}$$

We consider the *closed* Lie subgroup

$$K := \mathrm{Ad}^{-1}(\widehat{K})$$

of G : its Lie algebra is $\mathfrak{k}$. By definition K contains the center $Z = \mathrm{Ad}^{-1}(\mathrm{Id})$ of G. If we take the pull-back of (3.38) trough $\mathrm{Ad} : G \to \widehat{G}$ we get the diffeomorphism

$$\begin{array}{ccc} K \times \mathfrak{p} & \longrightarrow & G \\ (k, X) & \longmapsto & k \exp_G(X), \end{array} \tag{3.39}$$

which proves that K is connected since G is connected : hence K is the connected Lie subgroup of G associated to the Lie subalgebra $\mathfrak{k}$. Finally Z

belongs to K and $K/Z \simeq \widehat{K}$ is compact: the point (a), (b), (c) and (d) are proved.

Let $\Theta : G \to G$ defined by $\Theta(k \exp_G(X)) = k \exp_G(-X)$ for $k \in K$ and $X \in \mathfrak{p}$. We have obviously $\Theta^2 = 1$ and $\mathrm{Ad}(\Theta(g)) = {}^t\mathrm{Ad}(g)^{-1}$. If we take g_1, g_2 in G we see that

$$\begin{aligned}\mathrm{Ad}(\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1}) &= \left({}^t(\mathrm{Ad}(g_1)\mathrm{Ad}(g_2))^{-1}\right)\left({}^t\mathrm{Ad}(g_2)^{-1}\right){}^t\left(\mathrm{Ad}(g_1)^{-1}\right)\\ &= 1.\end{aligned}$$

So $\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1}) \in Z$ for every g_1, g_2 in G. Since G is connected and Z is discrete it gives $\Theta(g_1g_2)\Theta(g_2)^{-1}\Theta(g_1)^{-1} = 1$: (e) and (f) are proved. □

4 Invariant connections

A connection ∇ on the tangent bundle $\mathbf{T}M$ of a manifold M is a differential linear operator

$$\nabla : \Gamma(\mathbf{T}M) \longrightarrow \Gamma(\mathbf{T}^*M \otimes \mathbf{T}M) \tag{4.40}$$

satisfying th Leibnitz's rule: $\nabla(fs) = df \otimes s + f\nabla s$ for every $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(\mathbf{T}M)$. Here $\Gamma(-)$ denotes the space of sections of the corresponding bundle. The contraction of ∇s by $v \in \Gamma(\mathbf{T}M)$ is a vectors field on M denoted $\nabla_v s$.

The *torsion* of a connection ∇ on $\mathbf{T}M$ is the $(2,1)$-tensor T^∇ defined by

$$T^\nabla(u, v) = \nabla_u v - \nabla_v u - [u, v], \tag{4.41}$$

for all vectors fields u, v on M. The *curvature* of a connection ∇ on $\mathbf{T}M$ is the $(3,1)$-tensor R^∇ defined by

$$R^\nabla(u, v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}, \tag{4.42}$$

for all vectors fields u, v on M. Here $R^\nabla(u, v)$ is a differential operator acting on $\Gamma(\mathbf{T}M)$ which commutes with the multiplication by functions on M : so it is defined by the action of an element of $\Gamma(\mathrm{End}(\mathbf{T}M))$. For convenience we denote $R^\nabla(u, v) \in \Gamma(\mathrm{End}(\mathbf{T}M))$ this element. We can specialize the curvature tensor R^∇ at each $m \in M$: $R^\nabla_m(U, V) \in \mathrm{End}(\mathbf{T}_mM)$ for each $U, V \in \mathbf{T}_mM$.

4.1 Connections invariant under a group action

Suppose now that a lie group G acts smoothly on a manifold M. The corresponding action of G on the vectors spaces $\mathcal{C}^\infty(M)$, $\Gamma(\mathbf{T}M)$ and $\Gamma(\mathbf{T}^*M)$

is

$$\underline{g} \cdot f(m) = f(g^{-1}m), \quad m \in M,$$

$$\underline{g} \cdot s(m) = \mathbf{T}_{g^{-1}m} g(s(g^{-1}m)), \quad m \in M,$$

and

$$\underline{g} \cdot \xi(m) = \xi(g^{-1}m) \circ \mathbf{T}_m g^{-1}, \quad m \in M,$$

for every $f \in \mathcal{C}^\infty(M)$, $s \in \Gamma(\mathbf{T}M))$, $\xi \in \Gamma(\mathbf{T}^*M)$ and $g \in G$. Here we denote $\mathbf{T}_n g$ the differential at $n \in M$ of the smooth map $m \mapsto gm$. Note that the G-action is compatible with the canonical bracket $\langle -, - \rangle : \Gamma(\mathbf{T}^*M) \times \Gamma(\mathbf{T}M) \to \mathcal{C}^\infty(M)$: $\langle \underline{g} \cdot \xi, \underline{g} \cdot s \rangle = \underline{g} \cdot \langle \xi, s \rangle$. We still denote $\underline{g}$ the action of $g \in G$ on $\Gamma(\mathbf{T}^*M \otimes \mathbf{T}M)$.

Definition 4.1 *A connection ∇ on the tangent bundle $\mathbf{T}M$ is G-invariant if*

$$\underline{g} \nabla \underline{g}^{-1} = \nabla, \quad \text{for every } g \in G. \tag{4.43}$$

This condition is equivalent to asking that $\nabla_{\underline{g} \cdot v}(\underline{g} \cdot s) = \underline{g} \cdot (\nabla_v s)$ for every vectors fields s, v on M and $g \in G$.

For every $X \in \mathfrak{g}$, the differential of $t \to \exp_G(tX)$ at $t = 0$ defines linear operators on $\mathcal{C}^\infty(M)$, $\Gamma(\mathbf{T}M)$ and $\Gamma(\mathbf{T}^*M)$, all denoted $\mathcal{L}(X)$. For $f \in \mathcal{C}^\infty(M)$ and $s \in \Gamma(M)$ we have $\mathcal{L}(X)f = X_M(f)$ and $\mathcal{L}(X)s = [X_M, s]$ where X_M is the vectors field on M defined at Section 2.4. The map $X \mapsto \mathcal{L}(X)$ is a Lie algebra morphism :

$$[\mathcal{L}(X), \mathcal{L}(Y)] = \mathcal{L}([X, Y]), \quad \text{for all} \quad X, Y \in \mathfrak{g}. \tag{4.44}$$

Definition 4.2 *The* moment *of a G-invariant connection ∇ on $\mathbf{T}M$ is the linear endomorphism of $\Gamma(\mathbf{T}M)$ defined by*

$$\Lambda(X) = \mathcal{L}(X) - \nabla_{X_M}, \quad X \in \mathfrak{g}. \tag{4.45}$$

Since the $\Lambda(X)$, $X \in \mathfrak{g}$ commute with the multiplication by functions on M, we can and we will see the $\Lambda(X)$ as element of $\Gamma(\mathrm{End}(\mathbf{T}M))$. The invariance condition (4.43) tells us that the map $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ is G-equivariant:

$$\Lambda(\mathrm{Ad}(g)Y) = \underline{g} \Lambda(Y) \underline{g}^{-1}, \quad \text{for every} \quad (g, Y) \in G \times \mathfrak{g}. \tag{4.46}$$

If we differentiate (4.46) at $g = 1$, we get

$$\Lambda([X, Y]) = [\mathcal{L}(X), \Lambda(Y)], \quad \text{for every } X, Y \in \mathfrak{g}. \tag{4.47}$$

We finish this section by computing the values of the torsion and curvature on vectors fields generated by the G-action. A direct computation gives

$$T^\nabla(X_M, Y_M) = [X, Y]_M - \Lambda(X)Y_M + \Lambda(Y)X_M. \tag{4.48}$$

for every $X, Y \in \mathfrak{g}$. Now using (4.44) and (4.47) we have for the curvature

$$R^\nabla(X_M, Y_M) = [\Lambda(X), \Lambda(Y)] - \Lambda([X, Y]), \tag{4.49}$$

for every $X, Y \in \mathfrak{g}$.

4.2 Invariant Levi-Civita connections

Suppose now that the manifold M carries a Riemannian structure invariant under the Lie group G. The scalar product of two vectors fields u, v is just denote (u, v). The invariance condition is that the equality

$$\underline{g} \cdot (u, v) = (\underline{g} \cdot u, \underline{g} \cdot v) \tag{4.50}$$

holds in $\mathcal{C}^\infty(M)$ for $u, v \in \Gamma(\mathbf{T}M)$ and $g \in G$. If we differentiate (4.50) at $g = e$ we get

$$X_M(u, v) = ([X_M, u], v) + (u, [X_M, v]). \tag{4.51}$$

Let ∇^{LC} the Levi-Civita connection on M relative to the Riemannian metric: it is the unique torsion free connection which preserve the Riemannian metric. Since the Riemannian metric is G-invariant, the connection $\underline{g}\nabla^{\mathrm{LC}}\underline{g}^{-1}$ preserves also the Riemannian metric and is torsion free for every $g \in G$. Hence ∇^{LC} is a G-invariant connection. Recall that for $u, v \in \Gamma(\mathbf{T}M)$ the vectors field $\nabla^{\mathrm{LC}}_u v$ is defined by the relations

$$2(\nabla^{\mathrm{LC}}_u v, w) = ([u, v], w) - ([v, w], u) + ([w, u], v) + u(v, w) + v(u, w) - w(u, v). \tag{4.52}$$

If we take $u = X_M$ and $v = Y_M$ in the former relation we find with the help of (4.51) that

$$2(\nabla^{\mathrm{LC}}_{X_M} Y_M, w) = ([X, Y]_M, w) - w(X_M, Y_M). \tag{4.53}$$

So we have proved the

Proposition 4.3 *For any $X, Y \in \mathfrak{g}$ we have*

$$\nabla^{\mathrm{LC}}_{X_M} Y_M = \frac{1}{2}\Big([X, Y]_M - \overrightarrow{\mathrm{grad}}(X_M, Y_M).\Big)$$

5 Invariant connections on homogeneous spaces

The main references here are [2] and [4].

5.1 Existence of invariant connections

We work here with the homogeneous space $M = G/H$ where H is a closed subgroup with Lie algebra $\mathfrak{h}$ of a Lie group G. We denote by $\pi : G \to M$ the quotient map. The quotient vector space $\mathfrak{g}/\mathfrak{h}$ is equipped with the H-action induced by the adjoint action. We consider the space $G \times \mathfrak{g}/\mathfrak{h}$ with the following H-action: $h \cdot (g, \overline{X}) = (gh^{-1}, \overline{\mathrm{Ad}(h)X})$. This action is proper and free so the quotient $G \times_H \mathfrak{g}/\mathfrak{h}$ is a smooth manifold: the class of $(g, \overline{X})$ in $G \times_H \mathfrak{g}/\mathfrak{h}$ is denoted $[g, \overline{X}]$. We use here the following G-equivariant isomorphism

$$\begin{array}{rcl} G \times_H \mathfrak{g}/\mathfrak{h} & \longrightarrow & \mathbf{T}M \\ {[g, \overline{X}]} & \longmapsto & \frac{d}{dt}\pi(g \exp_G(tX))|_{t=0}. \end{array} \tag{5.54}$$

Using the G-equivariant isomorphism (5.54) we have

$$\begin{array}{rcl} \Gamma(\mathbf{T}M) & \xrightarrow{\sim} & (\mathcal{C}^\infty(G) \otimes \mathfrak{g}/\mathfrak{h})^H \\ s & \mapsto & \widetilde{s} \end{array} \tag{5.55}$$

and

$$\begin{array}{rcl} \Gamma(\mathrm{End}(\mathbf{T}M)) & \xrightarrow{\sim} & (\mathcal{C}^\infty(G) \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h}))^H \\ A & \mapsto & \widetilde{A}. \end{array} \tag{5.56}$$

For example, the vectors field X_M, $X \in \mathfrak{g}$ give rise through the isomorphism (5.55) to the functions $\widetilde{X_M}(g) = -\mathrm{Ad}(g)^{-1}X \mod \mathfrak{g}/\mathfrak{h}$.

Let ∇ be a G-invariant connection on the tangent bundle $\mathbf{T}M$, and let $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ be the corresponding G-equivariant map defined by (4.45). Let $\widetilde{\Lambda} : \mathfrak{g} \to (\mathcal{C}^\infty(G) \otimes \mathrm{End}(\mathfrak{g}/\mathfrak{h}))^H$ be the map Λ through the identifications (5.56). The mapping $\widetilde{\Lambda}$ is G-equivariant and each $\widetilde{\Lambda}(X)$, $X \in \mathfrak{g}$ is a H-equivariant map from G to $\mathrm{End}(\mathfrak{g}/\mathfrak{h})$:

$$\begin{array}{rcl} \widetilde{\Lambda}(\mathrm{Ad}(g)X)(g') & = & \widetilde{\Lambda}(X)(g^{-1}g') \\ \widetilde{\Lambda}(X)(gh^{-1}) & = & \mathrm{Ad}(h) \circ \widetilde{\Lambda}(X)(g) \circ \mathrm{Ad}(h)^{-1} \end{array} \tag{5.57}$$

for every $g, g' \in G$, $h \in H$ and $X \in \mathfrak{g}$.

Definition 5.1 *Let* $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ *the map defined by* $\lambda(X) = \widetilde{\Lambda}(X)(e)$.

From (5.57), we see that λ is H-equivariant and determines completely Λ :

$$\widetilde{\Lambda}(X)(g) = \lambda(\mathrm{Ad}(g)^{-1}X). \tag{5.58}$$

So we have proved that the G-invariant connection ∇ is uniquely determined by the mapping $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$.

Proposition 5.2 *(a) The linear map* $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ *is* H*-equivariant, and when restrict to* $\mathfrak{h}$ *is equal to the adjoint action.*

(b) A linear map λ *satisfying the conditions of* (a) *determine a unique* G*-invariant connection on* $\mathbf{T}(G/H)$.

PROOF : We have $\Lambda(X) = \mathcal{L}(X) - \nabla_{X_M}$. So if $X_M(m) = 0$[6], we have $\Lambda(X)_m = \mathcal{L}(X)_m$ as endomorphism of $\mathbf{T}_m M$. When $m = \overline{e} \in M$, $X_M(\overline{e}) = 0$ if and only if $X \in \mathfrak{h}$, and then the endomorphism $\mathcal{L}(X)_{\overline{e}}$ of $\mathbf{T}_{\overline{e}}M = \mathfrak{g}/\mathfrak{h}$ is equal to $\mathrm{ad}(X)$. So $\lambda(X) = \mathrm{ad}(X)$ for all $X \in \mathfrak{h}$. The first point is proved.

Let $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}/\mathfrak{h})$ be a linear map satisfying the conditions (a), and let $\Lambda : \mathfrak{g} \to \Gamma(\mathrm{End}(\mathbf{T}M))$ be the corresponding G-equivariant map defined by λ : for $\overline{g} \in M$ and $X \in \mathfrak{g}$ the map $\Lambda(X)_{\overline{g}}$ is

$$\begin{array}{ccc} \mathbf{T}_{\overline{g}}M & \longrightarrow & \mathbf{T}_{\overline{g}}M \\ [\mathrm{g,Y}] & \longmapsto & [g, \lambda(g^{-1}X)Y]. \end{array}$$

By definition we have $\Lambda(X)_{\overline{g}} = \mathcal{L}(X)_{\overline{g}}$ when $X_M(\overline{g}) = 0$. Finally we define a G-invariant connection ∇ on $\mathbf{T}M$ by posing for any vectors field v, s on M and $m \in M$:

$$(\nabla_v s)|_m = (\mathcal{L}(X)s)|_m - \Lambda(X)_m(s|_m),$$

where $X \in \mathfrak{g}$ is chosen so that $X_M(m) = s|_m$. $\square$

COUNTER EXAMPLE : Consider the homogeneous space[7] $M = \mathrm{SL}(2,\mathbb{R})/H$ where

$$H = \{\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0\}.$$

We are going to prove that *the tangent bundle* $\mathbf{T}M$ *does not carry a* G*-invariant connection.* Consider the basis (e, f, g) of $\mathfrak{sl}(2,\mathbb{R})$, where

$$e = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

[6] $X_M(m) = 0$ if and only if m is fixed by the 1-parameter subgroup $\exp_G(\mathbb{R}X)$

[7] The manifold M is diffeomophic to the circle

We have $[e,f] = 2e$, $[g,f] = -2g$, and $[e,g] = -f$. Since the Lie algebra of H is $\mathfrak{h} := \mathbb{R}f \oplus \mathbb{R}g$, we use the identifications $\mathfrak{sl}(2,\mathbb{R})/\mathfrak{h} \cong \mathbb{R}e$ and $\mathrm{End}(\mathfrak{sl}(2,\mathbb{R})/\mathfrak{h}) \cong \mathbb{R}$. For the induced adjoint action of $\mathfrak{h}$ on $\mathbb{R}e$ we have : $\widehat{\mathrm{ad}}(f) = -2$ and $\widehat{\mathrm{ad}}(g) = 0$. We are interested in a map $\lambda : \mathfrak{sl}(2,\mathbb{R}) \to \mathbb{R}$ satisfying

- λ is H-equivariant, i.e. $\lambda([X,Y]) = 0$ whenever $X \in \mathfrak{h}$.
- $\lambda(X) = \widehat{\mathrm{ad}}(X)$ for $X \in \mathfrak{h}$.

Theses conditions can not be fullfilled since the first point gives $\lambda(f) = \lambda([g,e]) = 0$, and with the second point we have $\lambda(f) = \widehat{\mathrm{ad}}(f) = -2$.

The previous example shows that some homogeneous spaces do not have invariant connection. For the remaining of Section 5 we work with the following

Assumption 5.3 *The subalgebra $\mathfrak{h}$ has a H-invariant supplementary subspace $\mathfrak{m}$ in $\mathfrak{g}$.*

In [4] the homogeneous spaces G/H are called of *reductive type* when the assumption 5.3 is satisfied. This hypothesis garanties the existence of invariant connections as we will see now.

Let $X \mapsto X_{\mathfrak{m}}$ denotes the H-equivariant projection onto $\mathfrak{m}$ relatively to $\mathfrak{h}$. This projection induces an H-equivariant isomorphism $\mathfrak{g}/\mathfrak{h} \simeq \mathfrak{m}$. Then a G-invariant connection on $\mathbf{T}(G/H)$ is determined uniquely by a linear H-equivariant mapping $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ which extends the adjoint action $\mathrm{ad} : \mathfrak{h} \to \mathrm{End}(\mathfrak{m})$. So λ is completely determined by its restriction

$$\lambda|_{\mathfrak{m}} : \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$$

The following definition defines a family ∇^a, $a \in \mathbb{R}$ of invariant connection when G/H is an homogeneous spaces of *reductive type*.

Definition 5.4 *Let G/H be an homogeneous spaces of* reductive type. *For any $a \in \mathbb{R}$, we define a H-equivariant mapping $\lambda^a : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ by $\lambda^a(X) = \mathrm{ad}(X)$ for $X \in \mathfrak{h}$ and*

$$\lambda^a(X)Y = a[X,Y]_{\mathfrak{m}} \quad \text{for} \quad X, Y \in \mathfrak{m}.$$

We denote ∇^a the G-invariant connection associated to λ^a.

The connection ∇^0 is called the *canonical* connection. Note that the connections $\nabla^a, a \in \mathbb{R}$ are distincts except when the bracket $[-,-]_{\mathfrak{m}} = 0$ is identically equal to 0.

We finish this section by looking to the torsion free invariant connections.

Proposition 5.5 *Let ∇ be a G-invariant connection on $\mathbf{T}(G/H)$ and let $\lambda : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ be the associated H-equivariant map. The connection ∇ is* torsion free *if and only if we have*

$$[X,Y]_{\mathfrak{m}} = \lambda(X)Y - \lambda(Y)X \quad \text{for all} \quad X,Y \in \mathfrak{m}. \tag{5.59}$$

Condition (5.59) is equivalent to asking that

$$\lambda(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + b(X,Y), \tag{5.60}$$

where $b : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$ is a symmetric *bilinear map.*

PROOF : The vectors fields X_M, $X \in \mathfrak{g}$ generates the tangent space of $M = G/H$, hence the connection is torsion free if and only if $T^{\nabla}(X_M, Y_M) = 0$ for every $X, Y \in \mathfrak{g}$. Following (4.48) the condition is

$$[X,Y]_M = \Lambda(X)Y_M - \Lambda(Y)X_M \quad \text{for all} \quad X,Y \in \mathfrak{g}. \tag{5.61}$$

A small computations shows that the function $\widetilde{X_M} : G \to \mathfrak{m}$ associated to the vectors field X_M via the isomorphism (5.55) is defined by $\widetilde{X_M}(g) = -[\mathrm{Ad}(g)^{-1}X]_{\mathfrak{m}}$. For the function $\widetilde{\lambda(X)Y_M} : G \to \mathfrak{m}$ we have

$$\widetilde{\lambda(X)Y_M}(g) = -\lambda(\mathrm{Ad}(g)^{-1}X)[\mathrm{Ad}(g)^{-1}Y]_{\mathfrak{m}}, \quad \text{for all} \quad X,Y \in \mathfrak{g}.$$

So condition (5.61) is equivalent to

$$[X,Y]_{\mathfrak{m}} = \lambda(X)Y_{\mathfrak{m}} - \lambda(Y)X_{\mathfrak{m}} \quad \text{for all} \quad X,Y \in \mathfrak{g}. \tag{5.62}$$

It is now easy to see that (5.62) is equivalent to (5.59) and (5.60). $\Box$

Corollary 5.6 *Let ∇^a be the G-invariant connection introduced in Definition 5.4. After Proposition 5.5, we see that*

- *if the bracket $[-,-]_{\mathfrak{m}}$ is identically equal to 0 : $\nabla^a = \nabla^0$ is torsion free.*
- *if the bracket $[-,-]_{\mathfrak{m}}$ is not equal to 0, ∇^a is torsion free if and only if $a = \frac{1}{2}$.*

5.2 Geodesics on an homogeneous spaces

Let ∇ be a G-invariant connection on $M = G/H$ associated to a H-equivariant map $\lambda : \mathfrak{m} \to \mathrm{End}(\mathfrak{m})$. A smooth curve $\gamma : I \to M$ is a *geodesic* relarive to a ∇ if

$$\nabla_{\gamma'}(\gamma') = 0. \tag{5.63}$$

The last condition can be understood locally as follow. Let $t_0 \in I$ and let $\mathcal{U} \subset M$ be a neighborhood of $\gamma(t_0)$: if $\mathcal{U}$ is small enough there exists a vectors field v on $\mathcal{U}$ such that $v(\gamma(t)) = \gamma'(t)$ for $t \in I$ closed to t_0. Then for t near t_0, condition (5.63) is equivalent to

$$\nabla_v v|_{\gamma(t)} = 0. \tag{5.64}$$

Proposition 5.7 *For $X \in \mathfrak{m}$, we consider the curve $\gamma_X(t) = \pi(\exp_G(tX))$ on G/H, where $\pi : G \to G/H$ denotes the canonical projection and $\exp_G$ is the exponential map of the lie group G. The curve γ_X is a geodesic for the connection ∇, if and only if $\lambda(X)X = 0$.*

Proof : The vectors field X_M, which is defined on M, satisfies $X_M(\gamma_X(t)) = \gamma'_X(t)$ for $t \in \mathbb{R}$. Since $\nabla_{X_M} X_M = \Lambda(X)X_M$ we get

$$\nabla_{X_M} X_M|_{\gamma_X(t)} = [\gamma_X(t), \lambda(X)X] \quad \text{in} \quad \mathbf{T}M \simeq G \times_H \mathfrak{m},$$

so the conclusion follows. □

Corollary 5.8 *Let ∇^a be the connection on G/H defined in Def. (5.4). Then*

- *the maximal geodesic are the curves $\gamma(t) = \pi(g \exp_G(tX))$, where $g \in G$ and $X \in \mathfrak{m}$.*
- *the exponential mapping $\exp_{\overline{e}} : \mathfrak{m} \to G/H$ is defined by $\exp_{\overline{e}}(X) = \pi(\exp_G(X))$.*

5.3 Levi-civita connection on homogeneous spaces

We suppose now that one has a $\mathrm{Ad}(H)$-invariant scalar product on the supplementary subspace $\mathfrak{m}$ of $\mathfrak{h}$, that we just denote $(-,-)$.

We define a G-invariant Riemannian metric $(-,-)_M$ on $M = G/H$ as follows. Using the identification $G \times_H \mathfrak{m} \simeq \mathbf{T}M$, we take $(v, w)_M = (X, Y)$ for the tangent vector $v = [g, X]$ and $w = [g, Y]$ of $\mathbf{T}_{\overline{g}}M$. Let ∇^{LC} the Levi-Civita connection on M relative to this Riemannian metric. Since the Riemannian metric is G-invariant, the connection ∇^{LC} is G-invariant (see Section 4.2). Let $\lambda^{\mathrm{LC}} : \mathfrak{g} \to \mathrm{End}(\mathfrak{m})$ the H-equivariant map associated to the connection ∇^{LC}. Since ∇^{LC} preserves the metric we have

$$\lambda^{\mathrm{LC}}(X) \in \mathrm{so}(\mathfrak{m}) \quad \text{for every} \quad X \in \mathfrak{g}. \tag{5.65}$$

Here $\mathrm{so}(\mathfrak{m})$ denotes the Lie algebra of the orthogonal group $\mathrm{SO}(\mathfrak{m})$.

Proposition 5.9 *The map* $\lambda^{\rm LC}$ *is determined by the following conditions:* $\lambda^{\rm LC}(X) = {\rm ad}(X)$ *for* $X \in \mathfrak{h}$ *and* $\lambda^{\rm LC}(X)Y = \frac{1}{2}[X,Y]_{\mathfrak{m}} + b^{\rm LC}(X,Y)$ *for* $X, Y \in \mathfrak{m}$*, where* $b^{\rm LC} : \mathfrak{m}\times\mathfrak{m}\to\mathfrak{m}$ *is the* symmetric *bilinear map defined by*

$$2(b^{\rm LC}(X,Y),Z) = ([Z,X]_{\mathfrak{m}},Y) + ([Z,Y]_{\mathfrak{m}},X) \quad X,Y,Z\in\mathfrak{m}. \tag{5.66}$$

Proof : We uses the decomposition (5.60) together with the fact that $(\lambda^{\rm LC}(X)Y,Z) = -(Y,\lambda^{\rm LC}(X)Z)$ for $X,Y,Z\in\mathfrak{m}$. It gives

$$(b^{\rm LC}(X,Y),Z) + (b^{\rm LC}(Z,X),Y) = \frac{-1}{2}\Big(([X,Y]_{\mathfrak{m}},Z) + ([X,Z]_{\mathfrak{m}},Y)\Big). \tag{5.67}$$

Now if we interchange X,Y,Z in Z,X,Y and after in Y,Z,X, we get two other equalities. If we sum them with alternative sign we get on the LHS the term $2(b^{\rm LC}(X,Y),Z)$ and on the RHS we get $-([X,Z]_{\mathfrak{m}},Y) - ([Y,Z]_{\mathfrak{m}},X)$. □

Example. Suppose that G is a compact Lie group and H is a closed subgroup. Let $(-,-)_{\mathfrak{g}}$ be a G-invariant scalar product on $\mathfrak{g}$. We take $\mathfrak{m}$ as the orthogonal subspace of $\mathfrak{h}$. We take on G/H the G-invariant Riemannian metric coming from the scalar product $(-,-)_{\mathfrak{g}}$ restricted to $\mathfrak{m}$. In this situation we see that the bilinear map $b^{\rm LC}$ vanishes. So, the Levi-Civita connection on G/H is equal to the connection $\nabla^{1/2}$ (see Definition 5.4). Then we know after Corollary 5.8 that the geodesics on G/H are of the form $\gamma(t) = \pi(g\exp_G(tX))$ with $X\in\mathfrak{m}$.

5.4 Levi-civita connection on symmetric spaces of the non-compact type.

We come back to the situation of section 3.4. Let G be a connected semi-simple Lie group with algebra $\mathfrak{g}$. Let $\Theta : G\to G$ be an involution of G such that $\theta = d\Theta$ is a Cartan involution of $\mathfrak{g}$. At the Lie algebra level we have the decomposition $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ where $\mathfrak{k}$ is the Lie algebra of the closed connected subgroup $K = G^{\Theta}$ and $\mathfrak{p} = \{X\in\mathfrak{g} \mid \theta(X) = -X\}$. We denote by $X\mapsto X_{\mathfrak{k}}$ and $X\mapsto X_{\mathfrak{p}}$ the projections such that $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$ for $X\in\mathfrak{g}$.

We consider here the homogeneous space $M = G/K$. Since ${\rm Ad}(K)$ is compact, the vector subspace $\mathfrak{p}\simeq \mathbf{T}_{\bar{e}}M$ carries ${\rm Ad}(K)$-invariant scalar product that induces G-invariant Rieamannian metric on M. One of them is of particular interest : the Killing form $B_{\mathfrak{g}}$.

Proposition 5.10 *The Levi-Civita connection* $\nabla^{\rm LC}$ *on* G/K *associated to* any ${\rm Ad}(K)$*-invariant scalar product on* $\mathfrak{p}$ *coincides with the canonical connection* ∇^0 *(see Definition 5.4).*

PROOF : Since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$, we have $[X,Y]_{\mathfrak{p}} = 0$ when $X, Y \in \mathfrak{p}$. After Proposition 5.9, we have then $\lambda^{\mathrm{LC}}(X) = \mathrm{ad}(X_{\mathfrak{k}})$ for $X \in \mathfrak{p}$, which means that $\nabla^{\mathrm{LC}} = \nabla^0$. $\square$

In this setting Corollary 5.8 gives

Corollary 5.11 • *All the maximal geodesic on G/K are defined over $\mathbb{R}$: the Riemannian manifold G/K is completed.*

• *the exponential mapping* $\exp_{\bar{e}} : \mathfrak{p} \to G/K$ *is defined by* $\exp_{\bar{e}}(X) = \pi(\exp_G(X))$.

We will now compute the curvature tensor R^{LC} of ∇^{LC}. By definition R^{LC} is a 2-form on M with values in $\mathrm{End}(\mathbf{T}M)$. We take $X, Y \in \mathfrak{g}$ and look at $R^{\mathrm{LC}}(X_M, Y_M) \in \Gamma(\mathrm{End}(\mathbf{T}M))$ or equivalently at the function $\widetilde{R^{\mathrm{LC}}(X_M, Y_M)} : G \to \mathrm{End}(\mathfrak{p})$: (4.49) gives

$$\begin{aligned}
\widetilde{R^{\mathrm{LC}}(X_M, Y_M)}(g) &= -[\lambda^{\mathrm{LC}}(g^{-1}X), \lambda^{\mathrm{LC}}(g^{-1}X)] + \lambda^{\mathrm{LC}}([g^{-1}X, g^{-1}Y]) \\
&= -[\mathrm{ad}((g^{-1}X)_{\mathfrak{k}}), \mathrm{ad}((g^{-1}X)_{\mathfrak{k}})] + \mathrm{ad}([g^{-1}X, g^{-1}Y]_{\mathfrak{k}}) \\
&= \mathrm{ad}([(g^{-1}X)_{\mathfrak{p}}, (g^{-1}Y)_{\mathfrak{p}}]).
\end{aligned}$$

At the point $\bar{e} \in M$, the curvature tensor R^{LC} specializes in a map $R^{\mathrm{LC}}_{\bar{e}} : \mathfrak{p} \times \mathfrak{p} \to \mathrm{End}(\mathfrak{p})$.

Proposition 5.12 *For $X, Y \in \mathfrak{p}$, we have*

$$R^{\mathrm{LC}}_{\bar{e}}(X, Y) = \mathrm{ad}([X, Y]).$$

We will now compute the sectional curvature when the Riemannian metric on $M = G/K$ is induced by the scalar product on $\mathfrak{p}$ defined by the Killing form $B_{\mathfrak{g}}$. The sectional curvature is a real function κ defined on the Grassmannian $Gr_2(\mathbf{T}M)$ of 2-dimensional vector subspaces of $\mathbf{T}M$ (see []). If $S \subset \mathbf{T}_{\bar{e}}M$ is generated by two *orthogonal* vectors $X, Y \in \mathfrak{p}$ we have

$$\begin{aligned}
\kappa(S) &= \frac{B_{\mathfrak{g}}(R^{\mathrm{LC}}_{\bar{e}}(X,Y)X, Y)}{\|X\|^2\|Y\|^2} \quad [1] \\
&= \frac{B_{\mathfrak{g}}([[X,Y],X],Y)}{\|X\|^2\|Y\|^2} \quad [2] \\
&= -\frac{\|[X,Y]\|^2}{\|X\|^2\|Y\|^2} \quad [3].
\end{aligned}$$

[1] is the definition of the sectional curvature. [2] is due to Proposition 5.12, and [3] follows from the $\mathfrak{g}$-invariance of the Killing form and also to the fact that $-B_{\mathfrak{g}}$ defines a scalar product on $\mathfrak{k}$.

CONCLUSION : The homogeneous manifold G/K, when equipped with the Riemannian metric induced by the Killing form, is a completed Riemannian manifold with negative sectional curvature.

5.5 Flats on symmetric spaces of the non-compact type

Let M be a Riemmannian manifold and N a connected submanifold of M. The submanifold N is called *totally geodesic* if for each geodesic $\gamma : I \to M$ of M we have for $t_0 \in I$

$$\Big(\gamma(t_0) \in N \quad \text{and} \quad \gamma'(t_0) \in \mathbf{T}_{\gamma(t_0)}N\Big) \Longrightarrow \gamma(t) \in N \quad \text{for all} \quad t \in I.$$

We consider now the case of the symmetric space G/K equipped with the Levi-Civita connection ∇^0.

Theorem 5.13 *The set of totally geodesic submanifold of G/K containing $\bar{e}$ is in one to one correspondence with the subspaces*[8] $\mathfrak{s} \subset \mathfrak{p}$ *satisfying* $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{s}]] \subset \mathfrak{s}$.

For a proof see [2][Section IV.7]. The correspondence works as follows. If S is a totally geodesic submanifold of G/K, one has $R^{\mathrm{LC}}_n(u, v)w \in T_nS$ for each $n \in S$ and $u, v, w \in \mathbf{T}_nS$. Then when $\bar{e} \in S$ one takes $\mathfrak{s} := \mathbf{T}_{\bar{e}}S$: the last condition becomes $[[u, v], w] \in \mathfrak{s}$ for $u, v, w \in \mathfrak{s}$.

In the other direction, if $\mathfrak{s}$ is a Lie triple system one sees that $\mathfrak{g}_{\mathfrak{s}} := [\mathfrak{s}, \mathfrak{s}] \oplus \mathfrak{s}$ is a Lie subalgebra of $\mathfrak{g}$. Let $G_{\mathfrak{s}}$ be the connected Lie subgroup of G associated to $\mathfrak{g}_{\mathfrak{s}}$. One can prove that the orbit $S := G_{\mathfrak{s}} \cdot \bar{e}$ is a closed submanifold of G/K which is totally geodesic.

We are interested now in the "flats" of G/K. These are the totally geodesic submanifold with a curvature tensor that vanishes identically. If we use the last Theorem one sees that the set of flats in G/K passing through $\bar{e}$ is in one to one correspondence with the set of *abelian* subspaces of $\mathfrak{p}$.

We will conclude this section with the

Lemma 5.14 *Let $\mathfrak{s}, \mathfrak{s}'$ be two maximal abelian subspaces of $\mathfrak{p}$. Then there exists $k_o \in K$ such that* $\mathrm{Ad}(k_o)\mathfrak{s} = \mathfrak{s}'$. *In particular the subspaces $\mathfrak{s}$ and $\mathfrak{s}'$ have the same dimension.*

PROOF: FIRST STEP. Let us show that for any maximal abelian subspace $\mathfrak{s}$ there exists $X \in \mathfrak{s}$ such that the stabilizer $\mathfrak{g}^X := \{Y \in \mathfrak{g} \mid [X, Y] = 0\}$ satisfies $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{s}$. We look at the *commuting* family $\mathrm{ad}(X)$, $X \in \mathfrak{s}$ of

[8] Such subspace of $\mathfrak{p}$ are called Lie triple system.

linear map on $\mathfrak{g}$. All these maps are *symmetric* relative to the scalar product $B^\theta := -B_{\mathfrak{g}}(\cdot, \theta(\cdot))$, so they can be diagonalized all together : there exists a finite set $\alpha_1, \cdots, \alpha_r$ of non-zero linear maps from $\mathfrak{s}$ to $\mathbb{R}$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{i=1}^{r} \mathfrak{g}_{\alpha_i},$$

with $\mathfrak{g}_{\alpha_i} = \{X \in \mathfrak{g} \mid [Z, X] = \alpha_i(Z)X, \forall Z \in \mathfrak{s}\}$. Here the subspace $\mathfrak{s}$ belongs to $\mathfrak{g}_0 = \{X \in \mathfrak{g} \mid [Z, X] = 0, \forall Z \in \mathfrak{s}\}$. Since we have assume that $\mathfrak{s}$ is maximal abelian in $\mathfrak{p}$ we have $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$. For any $X \in \mathfrak{s}$ we have obviously

$$\mathfrak{g}^X = \mathfrak{g}_0 \oplus \sum_{\alpha_i(X)=0} \mathfrak{g}_{\alpha_i}.$$

If we take $X \in \mathfrak{s}$ such that $\alpha_i(X) \neq 0$ for all i, then $\mathfrak{g}^X = \mathfrak{g}_0$, hence $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{s}$.

SECOND STEP. Take $X \in \mathfrak{s}$ and $X' \in \mathfrak{s}'$ such that $\mathfrak{g}^X \cap \mathfrak{p} = \mathfrak{s}$ and $\mathfrak{g}^{X'} \cap \mathfrak{p} = \mathfrak{s}'$. We define the function $f(k) = B_{\mathfrak{g}}(X', \mathrm{Ad}(k)X),\ k \in K$. Let k_0 be a critical point of f (which exits since $\mathrm{Ad}(K)$ is compact). If we differentiate f at k_o we get $B_{\mathfrak{g}}(X', [Y, \mathrm{Ad}(k_o)X]) = 0,\ \forall\, Y \in \mathfrak{k}$. Since $B_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant we get $B_{\mathfrak{g}}([X', \mathrm{Ad}(k_o)X], Y) = 0,\ \forall\, Y \in \mathfrak{k}$, so $[X', \mathrm{Ad}(k_o)X] = 0$. Since $\mathfrak{g}^{\mathrm{Ad}(k_o)X} \cap \mathfrak{p} = \mathrm{Ad}(k_o)(\mathfrak{g}^X \cap \mathfrak{p}) = \mathrm{Ad}(k_o)\mathfrak{s}$, the last equality gives $X' \in \mathrm{Ad}(k_o)\mathfrak{s}$. And since $\mathrm{Ad}(k_o)\mathfrak{s}$ is an abelian subspace of $\mathfrak{p}$ we have then

$$\begin{aligned} \mathrm{Ad}(k_o)\mathfrak{s} &\subset \mathfrak{g}^{X'} \cap \mathfrak{p} \\ &\subset \mathfrak{s}'. \end{aligned}$$

Finally since $\mathfrak{s}, \mathfrak{s}'$ are two maximal abelian subspaces, the last equality insures that $\mathrm{Ad}(k_o)\mathfrak{s} = \mathfrak{s}'$. □

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